National Olympiad Second Round 2021

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Day 1 January 5th, 2022

1 Initially, one of the two boxes on the table is empty and the other contains 29 different colored marbles. By starting with the full box and performing moves in order, in each move, one or more marbles are selected from that box and transferred to the other box. At most, how many moves can be made without selecting the same set of marbles more than once?

2 If a polynomial with real coefficients of degree d has at least d coefficients equal to 1 and has d real roots, what is the maximum possible value of d?

(Note: The roots of the polynomial do not have to be different from each other.)

3 A circle Γ is tangent to the side BC of a triangle ABC at X and tangent to the side AC at Y. A point P is taken on the side AB. Let XP and YP intersect Γ at K and L for the second time, AK and BL intersect Γ at R and S for the second time. Prove that XR and YS intersect on AB.

Day 2 January 6th, 2022

4 Points D and E are taken on $[BC]$ and $[AC]$ of acute angled triangle ABC such that BD and CE are angle bisectors. Projections of D onto BC and BA are P and Q, projections of E onto CA and CB are R and S. Let $AP \cap CQ = X$, $AS \cap BR = Y$ and $BX \cap CY = Z$. Show that $AZ \perp BC$.

5 There are finitely many primes dividing the numbers $\{a \cdot b^n + c \cdot d^n : n = 1, 2, 3, ...\}$ where a, b, c, d are positive integers. Prove that $b = d$.

6 In a school, there are 2021 students, each having exactly k friends. There aren't three students such that all three are friends with each other. What is the maximum possible value of k?