Problems from the CMIMC 2021

www.artofproblemsolving.com/community/c2743703

by parmenides51, FlakeLCR

– Algebra & Number Theory

2.1 Find the unique 3 digit number $N = \underline{A}\ \underline{B}\ \underline{C}$, whose digits (A, B, C) are all nonzero, with the property that the product $P = \underline{A}\ \underline{B}\ \underline{C} \times \underline{A}\ \underline{B} \times \underline{A}$ is divisible by 1000.

Proposed by Kyle Lee

2.2 Suppose a, b are positive real numbers such that $a + a^2 = 1$ and $b^2 + b^4 = 1$. Compute $a^2 + b^2$.

Proposed by Thomas Lam

2.3 1.1 How many multiples of 12 divide $12!$ and have exactly 12 divisors?

Proposed by Adam Bertelli

2.4 What is the 101st smallest integer which can represented in the form $3^a + 3^b + 3^c$, where a, b, and c are integers?

Proposed by Dilhan Salgado

2.5 1.2 Suppose there are 160 pigeons and n holes. The 1st pigeon flies to the 1st hole, the 2nd pigeon flies to the 4th hole, and so on, such that the ith pigeon flies to the $(i^2 \bmod n)$th hole, where $k \bmod n$ is the remainder when k is divided by n. What is minimum n such that there is at most one pigeon per hole?

Proposed by Christina Yao

2.6 1.3 Let a and b be complex numbers such that $(a+1)(b+1) = 2$ and $(a^2+1)(b^2+1) = 32$. Compute the sum of all possible values of $(a^4 + 1)(b^4 + 1)$.

Proposed by Kyle Lee

2.7 For each positive integer n, let $\sigma(n)$ denote the sum of the positive integer divisors of n. How many positive integers $n \leq 2021$ satisfy

$$\sigma(3n) \geq \sigma(n) + \sigma(2n)?$$

Proposed by Kyle Lee

2.8 1.4 Let $f(x) = \frac{x^2}{8}$. Starting at the point $(7,3)$, what is the length of the shortest path that touches the graph of f, and then the x-axis?

Proposed by Sam Delatore

1.5 Suppose f is a degree 42 polynomial such that for all integers $0 \le i \le 42$,

$$f(i) + f(43+i) + f(2 \cdot 43 + i) + \cdots + f(46 \cdot 43 + i) = (-2)^i$$

Find $f(2021) - f(0)$.

Proposed by Adam Bertelli

1.6 Find the remainder when

$$\left\lfloor \frac{149^{151} + 151^{149}}{22499} \right\rfloor$$

is divided by 10^4.

Proposed by Vijay Srinivasan

1.7 As a gift, Dilhan was given the number $n = 1^1 \cdot 2^2 \cdots 2021^{2021}$, and each day, he has been dividing n by $2021!$ exactly once. One day, when he did this, he discovered that, for the first time, n was no longer an integer, but instead a reduced fraction of the form $\frac{a}{b}$. What is the sum of all distinct prime factors of b?

Proposed by Adam Bertelli

1.8 There are integers v, w, x, y, z and real numbers $0 \le \theta < \theta' \le \pi$ such that

$$\cos 3\theta = \cos 3\theta' = v^{-1}, \qquad w + x\cos\theta + y\cos 2\theta = z\cos\theta'.$$

Given that $z \neq 0$ and v is positive, find the sum of the 4 smallest possible values of v.

Proposed by Vijay Srinivasan

– Geometry

2.1 Triangle ABC has a right angle at A, $AB = 20$, and $AC = 21$. Circles ω_A, ω_B, and ω_C are centered at A, B, and C respectively and pass through the midpoint M of $\overline{BC}$. ω_A and ω_B intersect at $X \neq M$, and ω_A and ω_C intersect at $Y \neq M$. Find XY.

Proposed by Connor Gordon

2.2 1.1 Points A, B, and C lie on a line, in that order, with $AB = 8$ and $BC = 2$. B is rotated 20° counterclockwise about A to a point B', tracing out an arc R_1. C is then rotated 20° clockwise about A to a point C', tracing out an arc R_2. What is the area of the region bounded by arc R_1, segment $B'C$, arc R_2, and segment $C'B$?

Proposed by Thomas Lam

2.3 Consider trapezoid $[ABCD]$ which has $AB \parallel CD$ with $AB = 5$ and $CD = 9$. Moreover, $\angle C = 15^\circ$ and $\angle D = 75^\circ$. Let M_1 be the midpoint of AB and M_2 be the midpoint of CD. What is the distance M_1M_2?

Proposed by Daniel Li

2.4 A $2\sqrt{5}$ by $4\sqrt{5}$ rectangle is rotated by an angle θ about one of its diagonals. If the total volume swept out by the rotating rectangle is 62π, find the measure of θ in degrees.

Proposed by Connor Gordon

2.5 Emily is at $(0, 0)$, chilling, when she sees a spider located at $(1, 0)$! Emily runs a continuous path to her home, located at $(\sqrt{2} + 2, 0)$, such that she is always moving away from the spider and toward her home. That is, her distance from the spider always increases whereas her distance to her home always decreases. What is the area of the set of all points that Emily could have visited on her run home?

Proposed by Thomas Lam

2.6 1.2 In convex quadrilateral $ABCD$, $\angle ADC = 90^\circ + \angle BAC$. Given that $AB = BC = 17$, and $CD = 16$, what is the maximum possible area of the quadrilateral?

Proposed by Thomas Lam

2.7 1.3 Let $\triangle ABC$ be a triangle with $AB = 10$ and $AC = 16,$ and let I be the intersection of the internal angle bisectors of $\triangle ABC$. Suppose the tangents to the circumcircle of $\triangle BIC$ at B and C intersect at a point P with $PA = 8$. Compute the length of BC.

Proposed by Kyle Lee

2.8 1.4 Let $ABCDEF$ be an equilateral heaxagon such that $\triangle ACE \cong \triangle DFB$. Given that $AC = 7$, $CE = 8$, and $EA = 9$, what is the side length of this hexagon?

Proposed by Thomas Lam

1.5 Let $\gamma_1, \gamma_2, \gamma_3$ be three circles with radii $3, 4, 9,$ respectively, such that γ_1 and γ_2 are externally tangent at $C,$ and γ_3 is internally tangent to γ_1 and γ_2 at A and $B,$ respectively. Suppose the tangents to γ_3 at A and B intersect at $X.$ The line through X and C intersect γ_3 at two points, P and Q. Compute the length of PQ.

Proposed by Kyle Lee

1.6 Let circles ω and Γ, centered at O_1 and O_2 and having radii 42 and 54 respectively, intersect at points X, Y, such that $\angle O_1XO_2 = 105^\circ$. Points A, B lie on ω and Γ respectively such that $\angle O_1XA = \angle AXB = \angle BXO_2$ and Y lies on both minor arcs XA and XB. Define P to be the intersection of AO_2 and BO_1. Suppose XP intersects AB at C. What is the value of $\frac{AC}{BC}$?

Proposed by Puhua Cheng

1.7 Convex pentagon $ABCDE$ has $\overline{BC} = 17$, $\overline{AB} = 2\overline{CD}$, and $\angle E = 90^\circ$. Additionally, $\overline{BD} - \overline{CD} = \overline{AC}$, and $\overline{BD} + \overline{CD} = 25$. Let M and N be the midpoints of BC and AD respectively. Ray EA is extended out to point P, and a line parallel to AD is drawn through P, intersecting line EM at Q. Let G be the midpoint of AQ. Given that N and G lie on EM and PM respectively, and the perimeter of $\triangle QBC$ is 42, find the length of $\overline{EM}$.

Proposed by Adam Bertelli

1.8 Let ABC be a triangle with $AB < AC$ and ω be a circle through A tangent to both the B-excircle and the C-excircle. Let ω intersect lines AB, AC at X, Y respectively and X, Y lie outside of segments AB, AC. Let O be the center of ω and let OI_C, OI_B intersect line BC at J, K respectively. Suppose $KJ = 4$, $KO = 16$ and $OJ = 13$. Find $\frac{[KI_BI_C]}{[JI_BI_C]}$.

Proposed by Grant Yu

– Combinatorics & Computer Science

2.1 We have a 9 by 9 chessboard with 9 kings (which can move to any of 8 adjacent squares) in the bottom row. What is the minimum number of moves, if two pieces cannot occupy the same square at the same time, to move all the kings into an X shape (a 5×5 region where there are 5 kings along each diagonal of the X, as shown below)?

```
O       O
  O   O
    O
  O   O
O       O
```

Proposed by David Tang

2.2 Dilhan has objects of 3 types, A, B, and C, and 6 functions

$$f_{A,B}, f_{A,C}, f_{B,A}, f_{B,C}, f_{C,A}, f_{C,B}$$

where $f_{X,Y}$ takes in an object of type X and outputs an object of type Y. Dilhan wants to compose his 6 functions, without repeats, such that the resulting expression is well-typed, meaning an object can be taken in by the first function, and the resulting output can then be taken in by the second function, and so on. In how many orders can he compose his 6 functions, satisfying this constraint?

Proposed by Adam Bertelli

2.3 1.1 Adam has a box with 15 pool balls in it, numbered from 1 to 15, and picks out 5 of them. He then sorts them in increasing order, takes the four differences between each pair of adjacent balls, and finds exactly two of these differences are equal to 1. How many selections of 5 balls could he have drawn from the box?

Proposed by Adam Bertelli

2.4 Vijay has a stash of different size stones: in particular, he has 2021 types of stones, with sizes from 0 through 2020, and he has $2r+1$ stones of size r.

Vijay starts randomly (and without replacement) taking out stones from his stash and laying them out in a line. Vijay notices that the first stone of size 2020 comes before the first stone of size 2019, the first stone of size 2019 is before the first stone of size 2018, and so on. What is the probability of this happening?

Express your answer in terms of only basic arithmetic operations (division, exponentiation, etc.) and the factorial function.

Proposed by Misha Ivkov

2.5 Bill Gates and Jeff Bezos are playing a game. Each turn, a coin is flipped, and if Bill and Jeff have $m, n > 0$ dollars, respectively, the winner of the coin toss will take $\min(m, n)$ from the loser. Given that Bill starts with 20 dollars and Jeff starts with 21 dollars, what is the probability that Bill ends up with all of the money?

Proposed by Daniel Li

2.6 1.2 Adam is playing Minesweeper on a 9×9 grid of squares, where exactly $\frac{1}{3}$ (or 27) of the squares are mines (generated uniformly at random over all such boards). Every time he clicks on a square, it is either a mine, in which case he loses, or it shows a number saying how many of the (up to eight) adjacent squares are mines.

First, he clicks the square directly above the center square, which shows the number 4. Next, he clicks the square directly below the center square, which shows the number 1. What is the probability that the center square is a mine?

Proposed by Adam Bertelli

2.7 1.3 How many permutations of the string 0123456 are there such that no contiguous substrings of lengths $1 < \ell < 7$ have a sum of digits divisible by 7?

Proposed by Srinivasan Sathiamurthy

2.8 1.4 Suppose you have a 6 sided dice with 3 faces colored red, 2 faces colored blue, and 1 face colored green. You roll this dice 20 times and record the color that shows up on top. What is the expected value of the product of the number of red faces, blue faces, and green faces?

Proposed by Daniel Li

1.5 There are exactly 7 possible tetrominos (groups of 4 connected squares in a grid):

`https://cdn.discordapp.com/attachments/813077401265242143/816189385859006474/tetris.png`

Daniel has a 2×20210 rectangle and wants to tile the interior with tetrominos without overlaps, pieces sticking out, or extra pieces left over. Note that you are allowed to rotate tetrominos but not reflect them.

For how many multisets of tetrominos (ie. an ordered tuple of how many of each tile he has) is it possible to exactly tile his 2×20210 rectangle?

Proposed by Dilhan Salgado

1.6 Alice and Bob each flip 20 fair coins. Given that Alice flipped at least as many heads as Bob, what is the expected number of heads that Alice flipped?

Proposed by Adam Bertelli

1.7 How many non-decreasing tuples of integers $(a_1, a_2, \dots, a_{16})$ are there such that $0 \le a_i \le 16$ for all i, and the sum of all a_i is even?

Proposed by Nancy Kuang

1.8 An *augmentation* on a graph G is defined as doing the following:

- Take some set D of vertices in G, and duplicate each vertex $v_i \in D$ to create a new vertex v_i'.

- If there's an edge between a pair of vertices $v_i, v_j \in D$, create an edge between vertices v_i' and v_j'. If there's an edge between a pair of vertices $v_i \in D$, $v_j \notin D$, you can choose to create an edge between v_i' and v_j but do not have to.

A graph is called *reachable* from G if it can be created through some sequence of augmentations on G. Some graph H has n vertices and satisfies that both H and the complement of H are reachable from a complete graph of 2021 vertices. If the maximum and minimum values of n are M and m, find $M + m$.

Proposed by Oliver Hayman

– Team

---

1 Given a trapezoid with bases AB and CD, there exists a point E on CD such that drawing the segments AE and BE partitions the trapezoid into 3 similar isosceles triangles, each with long side twice the short side. What is the sum of all possible values of $\frac{CD}{AB}$?

Proposed by Adam Bertelli

---

2 Let $p_1, p_2, p_3, p_4, p_5, p_6$ be distinct primes greater than 5. Find the minimum possible value of

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - 6 \min(p_1, p_2, p_3, p_4, p_5, p_6)$$

Proposed by Oliver Hayman

---

3 Evaluate

$$\sum_{i=0}^{\infty} \frac{7^i}{(7^i+1)(7^i+7)}$$

Proposed by Connor Gordon

---

4 How many four-digit positive integers $\overline{a_1a_2a_3a_4}$ have only nonzero digits and have the property that $|a_i - a_j| \neq 1$ for all $1 \leq i < j \leq 4$?

Proposed by Kyle Lee

---

5 Let N be the fifth largest number that can be created by combining 2021 1's using addition, multiplication, and exponentiation, in any order (parentheses are allowed). If $f(x) = \log_2(x)$, and k is the least positive integer such that $f^k(N)$ is not a power of 2, what is the value of $f^k(N)$?

(Note: $f^k(N) = f(f(\cdots(f(N))))$, where f is applied k times.)

Proposed by Adam Bertelli

---

6 Let $P(x), Q(x)$, and $R(x)$ be three monic quadratic polynomials with only real roots, satisfying

$$P(Q(x)) = (x-1)(x-3)(x-5)(x-7)$$

$$Q(R(x)) = (x-2)(x-4)(x-6)(x-8)$$

for all real numbers x. What is $P(0) + Q(0) + R(0)$?

Proposed by Kyle Lee

---

7 Let P and Q be fixed points in the Euclidean plane. Consider another point O_0. Define O_{i+1} as the center of the unique circle passing through O_i, P and Q. (Assume that O_i, P, Q are never collinear.) How many possible positions of O_0 satisfy that $O_{2021} = O_0$?

Proposed by Fei Peng

---

8 Determine the number of functions f from the integers to $\{1, 2, \cdots, 15\}$ which satisfy

$$f(x) = f(x + 15)$$

and

$$f(x + f(y)) = f(x - f(y))$$

for all x, y.

Proposed by Vijay Srinivasan

---

9 Let ABC be a triangle with circumcenter O. Additionally, $\angle BAC = 20^\circ$ and $\angle BCA = 70^\circ$. Let D, E be points on side AC such that BO bisects $\angle ABD$ and BE bisects $\angle CBD$. If P and Q are points on line BC such that DP and EQ are perpendicular to AC, what is $\angle PAQ$?

Proposed by Daniel Li

---

10 How many functions $f : \{1, 2, 3, \ldots, 7\} \to \{1, 2, 3, \ldots, 7\}$ are there such that the set $\mathcal{F} = \{f(i) : i \in \{1, \ldots, 7\}\}$ has cardinality four, while the set $\mathcal{G} = \{f(f(f(i))) : i \in \{1, \ldots, 7\}\}$ consists of a single element?

Proposed by Sam Delatore

---

11 The set of all points (x, y) in the plane satisfying $x < y$ and $x^3 - y^3 > x^2 - y^2$ has area A. What is the value of A?

Proposed by Adam Bertelli

---

12 Let $\triangle ABC$ be a triangle, and let l be the line passing through its incenter and centroid. Assume that B and C lie on the same side of l, and that the distance from B to l is twice the distance from C to l. Suppose also that the length BA is twice that of CA. If $\triangle ABC$ has integer side lengths and is as small as possible, what is $AB^2 + BC^2 + CA^2$?

Proposed by Thomas Lam

---

13 Let $p = 3 \cdot 10^{10} + 1$ be a prime and let p_n denote the probability that $p \mid (k^k - 1)$ for a random k chosen uniformly from $\{1, 2, \cdots, n\}$. Given that $p_n \cdot p$ converges to a value L as n goes to infinity, what is L?

Proposed by Vijay Srinivasan

---

14 Let S be the set of lattice points $(x, y) \in \mathbb{Z}^2$ such that $-10 \le x, y \le 10$. Let the point $(0, 0)$ be O. Let Scotty the Dog's position be point P, where initially $P = (0, 1)$. At every second, consider all pairs of points $C, D \in S$ such that neither C nor D lies on line OP, and the area of quadrilateral $OCPD$ (with the points going clockwise in that order) is 1. Scotty finds the pair C, D maximizing the sum of the y coordinates of C and D, and randomly jumps to one of them, setting that as the new point P. After 50 such moves, Scotty ends up at point $(1,1)$. Find the probability that he never returned to the point $(0,1)$ during these 50 moves.

Proposed by David Tang

---

15 Adam has a circle of radius 1 centered at the origin.

- First, he draws 6 segments from the origin to the boundary of the circle, which splits the upper (positive y) semicircle into 7 equal pieces.

- Next, starting from each point where a segment hit the circle, he draws an altitude to the x-axis.

- Finally, starting from each point where an altitude hit the x-axis, he draws a segment directly away from the bottommost point of the circle $(0,-1)$, stopping when he reaches the boundary of the circle.

What is the product of the lengths of all 18 segments Adam drew?

`https://cdn.discordapp.com/attachments/813077401265242143/816190774257516594/circle2.png`

Proposed by Adam Bertelli

– Theoretical Computer Science

---

1 You place n^2 indistinguishable pieces on an $n \times n$ chessboard, where $n = 2^{90} \approx 1.27 \times 10^{27}$. Of these pieces, n of them are slightly lighter than usual, while the rest are all the same standard weight, but you are unable to discern this simply by feeling the pieces.

However, beneath each row and column of the chessboard, you have set up a scale, which, when turned on, will tell you *only* whether the average weight of all the pieces on that row or column is the standard weight, or lighter than standard. On a given step, you are allowed to rearrange every piece on the chessboard, and then turn on all the scales simultaneously, and record their outputs, before turning them all off again. (Note that you can only turn on the scales if all n^2 pieces are placed in different squares on the board.)

Find an algorithm that, in at most k steps, will always allow you to rearrange the pieces in such a way that every row and column measures lighter than standard on the final step.

An algorithm that completes in at most k steps will be awarded:

1 pt for $k > 10^{55}$
10 pts for $k = 10^{55}$
30 pts for $k = 10^{30}$
50 pts for $k = 10^{28}$

65 pts for $k = 10^{20}$
80 pts for $k = 10^5$
90 pts for $k = 2021$
100 pts for $k = 500$

---

2 You are initially given the number $n = 1$. Each turn, you may choose any positive divisor $d \mid n$, and multiply n by $d+1$. For instance, on the first turn, you must select $d = 1$, giving $n = 1 \cdot (1+1) = 2$ as your new value of n. On the next turn, you can select either $d = 1$ or 2, giving $n = 2 \cdot (1+1) = 4$ or $n = 2 \cdot (2+1) = 6$, respectively, and so on.

Find an algorithm that, in at most k steps, results in n being divisible by the number $2021^{2021^{2021}} - 1$.

An algorithm that completes in at most k steps will be awarded:

1 pt for $k > 2021^{2021^{2021}}$
20 pts for $k = 2021^{2021^{2021}}$
50 pts for $k = 10^{10^4}$
75 pts for $k = 10^{10}$
90 pts for $k = 10^5$
95 pts for $k = 6 \cdot 10^4$
100 pts for $k = 5 \cdot 10^4$

---

3 There is a tiger (which is treated as a point) in the plane that is trying to escape. It starts at the origin at time $t = 0$, and moves continuously at some speed k. At every positive integer time t, you can place one closed unit disk anywhere in the plane, so long as the disk does not intersect the tiger's current position. The tiger is not allowed to move into any previously placed disks (i.e. the disks block the tiger from moving). Note that when you place the disks, you can "see" the tiger (i.e. know where the tiger currently is).

Your goal is to prevent the tiger from escaping to infinity. In other words, you must show there is some radius $R(k)$ such that, using your algorithm, it is impossible for a tiger with speed k to reach a distance larger than $R(k)$ from the origin (where it started).

Find an algorithm for placing disks such that there exists some finite real $R(k)$ such that the tiger will never be a distance more than $R(k)$ away from the origin.

An algorithm that can trap a tiger of speed k will be awarded:

1 pt for $k < 0.05$
10 pts for $k = 0.05$
20 pts for $k = 0.2$
30 pts for $k = 0.3$
50 pts for $k = 1$
70 pts for $k = 2$
80 pts for $k = 2.3$

85 pts for $k = 2.6$
90 pts for $k = 2.9$
100 pts for $k = 3.9$
