Saudi Arabia IMO Team Selection Test 2010

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– Day I

1 Let $A_1A_2...A_{2010}$ be a regular 2010-gon. Find the number of obtuse triangles whose vertices are among A_1, A_2,..., A_{2010}.

2 Find all functions $f,g : N \to N$ such that for all $m,n \in N$ the following relation holds:

$$f(m) - f(n) = (m - n)(g(m) + g(n))$$

.

Note: $N = \{0, 1, 2, ...\}$

3 Consider a circle of center O and a chord AB of it (not a diameter). Take a point T on the ray OB. The perpendicular at T onto OB meets the chord AB at C and the circle at D and E. Denote by S the orthogonal projection of T onto the chord AB. Prove that $AS \cdot BC = TE \cdot TD$.

– Day II

1 Let ABC be a triangle with $\angle B \ge 2\angle C$. Denote by D the foot of the altitude from A and by M be the midpoint of BC. Prove that $DM \ge \frac{AB}{2}$.

2 The squares $OABC$ and $OA_1B_1C_1$ are situated in the same plane and are directly oriented. Prove that the lines AA_1 , BB_1, and CC_1 are concurrent.

3 Let $f : N \to N$ be a strictly increasing function such that $f(f(n)) = 3n$, for all $n \in N$. Find $f(2010)$.

Note: $N = \{0, 1, 2, ...\}$

– Day III

1 Find all real numbers x that can be written as

$$x = \frac{a_0}{a_1a_2..a_n} + \frac{a_1}{a_2a_3..a_n} + \frac{a_2}{a_3a_4..a_n} + ... + \frac{a_{n-2}}{a_{n-1}a_n} + \frac{a_{n-1}}{a_n}$$

where $n, a_1, a_2, ..., a_n$ are positive integers and $1 = a_0 \le a_1 < ... < a_n$

2 Points M and N are considered in the interior of triangle ABC such that $\angle MAB = \angle NAC$ and $\angle MBA = \angle NBC$. Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1$$

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3 Consider the arithmetic sequence $8, 21, 34, 47, ...$
a) Prove that this sequence contains infinitely many integers written only with digit 9.
b) How many such integers less than 2010^{2010} are in the sequence?

– Day IV

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1 Find all pairs (m, n) of integers, $m, n \geq 2$ such that $mn - 1$ divides $n^3 - 1$.

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2 Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle ADC = 135^o$ and

$$AC^2BD^2 = 2AB \cdot BC \cdot CD \cdot DA.$$

Prove that the diagonals of $ABCD$ are perpendicular.

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3 Consider the sequence $a_1 = 3$ and $a_{n+1} = \frac{3a_n^2+1}{2} - a_n$ for $n = 1, 2, ...$
Prove that if n is a power of 3 then n divides a_n .

– Day V

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1 In triangle ABC the circumcircle has radius R and center O and the incircle has radius r and center $I \neq O$. Let G denote the centroid of triangle ABC. Prove that $IG \perp BC$ if and only if $AB = AC$ or $AB + AC = 3BC$.

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2 a) Prove that for each positive integer n there is a unique positive integer a_n such that

$$(1 + \sqrt{5})^n = \sqrt{a_n} + \sqrt{a_n + 4^n}.$$

b) Prove that a_{2010} is divisible by $5 \cdot 4^{2009}$ and find the quotient

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3 Find all primes p for which $p^2 - p + 1$ is a perfect cube.
