Saudi Arabia Team Selection Test for Balkan Math Olympiad 2010

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–	Day I
1	Find all triples (x, y, z) of real numbers such that $x^2 + y^2 + z^2 + 1 = xy + yz + zx + \|x - 2y + z\|$.
2	Let ABC be an acute triangle and let $MNPQ$ be a square inscribed in the triangle such that $M, N \in BC$, $P \in AC$, $Q \in AB$. Prove that $area\,[MNPQ] \le \frac{1}{2} area\,[ABC]$.
3	Let $(a_n)_{n\ge o}$ and $(b_n)_{n\ge o}$ be sequences defined by $a_{n+2} = a_{n+1} + a_n$, $n = 0, 1, ...$, $a_0 = 1$, $a_1 = 2$, and $b_{n+2} = b_{n+1} + b_n$, $n = 0, 1, ...$, $b_0 = 2$, $b_1 = 1$. How many integers do the sequences have in common?
4	Let $f : N \to [0, \infty)$ be a function satisfying the following conditions: a) $f(4) = 2$ b) $\frac{1}{f(0)+f(1)} + \frac{1}{f(1)+f(2)} + ... + \frac{1}{f(n)+f(n+1)} = f(n + 1)$ for all integers $n \ge 0$. Find $f(n)$ in closed form.
–	Day II
1	Find all pairs (x, y) of positive integers such that $x^2 + y^2 + 33^2 = 2010\sqrt{x - y}$.
2	Show that in any triangle ABC with $\angle A = 90^o$ the following inequality holds $$(AB - AC)^2(BC^2 + 4AB \cdot AC)^2 < 2BC^6.$$
3	Let $a > 0$ be a real number and let $f : R \to R$ be a function satisfying $$f(x_1) + f(x_2) \ge af(x_1 + x_2), \forall x_1, x_2 \in R.$$ Prove that $$f(x_1) + f(x_2) + (x_3) \ge \frac{3a^2}{a+2} f(x_1 + x_2 + x_3), \forall x_1, x_2, x_3 \in R$$.

4 In quadrilateral $ABCD$, diagonals AC and BD intersect at O. Denote by P, Q, R, S the orthogonal projections of O onto AB , BC ,CD , DA, respectively. Prove that

$$PA \cdot AB + RC \cdot CD = \frac{1}{2}(AD^2 + BC^2)$$

if and only if

$$QB \cdot BC + SD \cdot DA = \frac{1}{2}(AB^2 + CD^2)$$

– Day III

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1 Find all non-empty sets S of nonzero real numbers such that
a) S has at most 5 elements
b) If x is in S, then so are $1 - x$ and $\frac{1}{x}$.

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2 Consider a triangle ABC and a point P in its interior. Lines PA, PB, PC intersect BC, CA, AB at A', B', C' , respectively. Prove that

$$\frac{BA'}{BC} + \frac{CB'}{CA} + \frac{AC'}{AB} = \frac{3}{2}$$

if and only if at least two of the triangles PAB, PBC, PCA have the same area.

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3 Find all functions $f : R \to R$ such that

$$xf(x + xy) = xf(x) + f(x^2)f(y)$$

for all $x, y \in R$.

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4 Find all triples (x, y, z) of integers such that

$$\begin{cases} x^2y + y^2z + z^2x = 2010^2 \\ xy^2 + yz^2 + zx^2 = -2010 \end{cases}$$

– Day IV

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1 Find all triples (x, y, z) of positive integers such that $3^x + 4^y = 5^z$.

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2 Quadrilateral $ABCD$ with perpendicular diagonals AC and BD is inscribed in a circle. Altitude DE in triangle ABD intersects diagonal AC in F. Prove that $FB = BC$

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3 How many integers in the set $\{1, 2, ..., 2010\}$ divide $5^{2010!} - 3^{2010!}$?

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4 Let $a > 0$. If the system

$$\begin{cases} a^x + a^y + a^z = 14 - a \\ x + y + z = 1 \end{cases}$$

has a solution in real numbers, prove that $a \leq 8$.

– Day V

1 Find all integers n for which $9n + 16$ and $16n + 9$ are both perfect squares.

2 Evaluate the sum

$$1 + 2 + 3 - 4 - 5 + 6 + 7 + 8 - 9 - 10 + ... - 2010$$

, where each three consecutive signs $+$ are followed by two signs $-$.

3 Let ABC be a right angled triangle with $\angle A = 90^o$and $BC = a$, $AC = b$, $AB = c$. Let d be a line passing trough the incenter of triangle and intersecting the sides AB and AC in P and Q, respectively.
(a) Prove that

$$b \cdot \left(\frac{PB}{PA}\right) + c \cdot \left(\frac{QC}{QA}\right) = a$$

(b) Find the minimum of

$$\left(\frac{PB}{PA}\right)^2 + \left(\frac{QC}{QA}\right)^2$$

4 Find all primes p, q satisfying the equation $2p^q - q^p = 7$.