Saudi Arabia IMO Team Selection Test 2011

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– Day I

1 Let I be the incenter of a triangle ABC and let A', B', C' be midpoints of sides BC, CA, AB, respectively. If $IA' = IB' = IC'$, then prove that triangle ABC is equilateral.

2 Consider the set $S = \{(a+b)^7 - a^7 - b^7 : a, b \in Z\}$. Find the greatest common divisor of all members in S.

3 Let n be a positive integer. Prove that at least one of the integers $[2^n \cdot \sqrt{2}]$, $[2^{n+1} \cdot \sqrt{2}]$, ..., $[2^{2n} \cdot \sqrt{2}]$ is even, where $[a]$ denotes the integer part of a.

– Day II

1 Let a and b be integers such that $a - b = a^2c - b^2d$ for some consecutive integers c and d. Prove that $|a - b|$ is a perfect square.

2 Let ABC be a triangle with $AB \neq AC$. Its incircle has center I and touches the side BC at point D. Line AI intersects the circumcircle ω of triangle ABC at M and DM intersects again ω at P. Prove that $\angle API = 90^o$.

3 Find all functions $f : R \to R$ such that

$$2f(x) = f(x+y) + f(x+2y)$$

, for all $x \in R$ and for all $y \geq 0$.

– Day III

1 Let a, b, c be real numbers such that $ab + bc + ca = 1$. Prove that

$$\frac{(a+b)^2+1}{c^2+2} + \frac{(b+c)^2+1}{a^2+2} + \frac{(c+a)^2+1}{b^2+2} \geq 3$$

2 In triangle ABC, let I_a , I_b, I_c be the centers of the excircles tangent to sides BC, CA, AB, respectively. Let P and Q be the tangency points of the excircle of center I_a with lines AB and AC. Line PQ intersects I_aB and I_aC at D and E. Let A_1 be the intersection of DC and BE. In an analogous way we define points B_1 and C_1. Prove that AA_1, BB_1 , CC_1 are concurrent.

3 Let $f \in Z[X]$, $f = X^2 + aX + b$, be a quadratic polynomial. Prove that f has integer zeros if and only if for each positive integer n there is an integer u_n such that $n|f(u_n)$.

– Day IV

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1 Find all integers n, $n \geq 2$, such that the numbers $1!, 2!, ..., (n-1)!$ give distinct remainders when divided by n.

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2 Let ABC be a non-isosceles triangle with circumcenter O, incenter I, and orthocenter H. Prove that angle $\angle OIH$ is obtuse.

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3 In acute triangle ABC, $\angle A = 20^o$. Prove that the triangle is isosceles if and only if

$$\sqrt[3]{a^3 + b^3 + c^3 - 3abc} = \min\{b, c\}$$

, where a, b, c are the side lengths of triangle ABC.
