Saudi Arabia Team Selection Test for Balkan Math Olympiad 2011

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– Day I

1 Let n be a positive integer. Find all real numbers $x_1, x_2, ..., x_n$ such that

$$\prod_{k=1}^{n}(x_k^2+(k+2)x_k+k^2+k+1)=\left(\frac{3}{4}\right)^n(n!)^2$$

2 For any positive integer n, let a_n be the number of pairs (x, y) of integers satisfying $|x^2-y^2|=n$.
(a) Find a_{1432} and a_{1433}.
(b) Find a_n .

3 In an acute triangle ABC the angle bisector AL, $L\in BC$, intersects its circumcircle at N. Let K and M be the projections of L onto sides AB and AC. Prove that triangle ABC and quadrilateral $AKNM$ have equal areas.

4 Consider a non-zero real number a such that $\{a\}+\left\{\frac{1}{a}\right\}=1$, where $\{x\}$ denotes the fractional part of x. Prove that for any positive integer n, $\{a^n\}+\left\{\frac{1}{a^n}\right\}=1$.

– Day II

1 Find all polynomials P with real coefficients such that for all $x,y,z\in R$,

$$P(x)+P(y)+P(z)+P(x+y+z)=P(x+y)+P(y+z)+P(z+x)$$

2 For each positive integer n let the set A_n consist of all numbers $\pm1\pm2\pm...\pm n$. For example,

$$A_1=\{-1,1\}, A_2=\{-3,-1,1,3\}, A_3=\{-6,-4,-2,0,2,4,6\}.$$

Find the number of elements in A_n .

3 Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a+b+\frac{1}{abc}+1}+\frac{1}{b+c+\frac{1}{abc}+1}+\frac{1}{c+a+\frac{1}{abc}+1}\le\frac{a+b+c}{a+b+c+1}$$

4 Let ABC be a triangle with circumcenter O. Points P and Q are interior to sides CA and AB, respectively. Circle ω passes through the midpoints of segments BP, CQ, PQ. Prove that if line PQ is tangent to circle ω, then $OP = OQ$.

– Day III

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1 Let $ABCD$ be a square of center O. The parallel to AD through O intersects AB and CD at M and N and a parallel to AB intersects diagonal AC at P. Prove that

$$OP^4 + \left(\frac{MN}{2}\right)^4 = MP^2 \cdot NP^2$$

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2 Let n be a positive integer. Prove that all roots of the equation

$$x(x+2)(x+4)...(x+2n) + (x+1)(x+3)...(x+2n-1) = 0$$

are real and irrational.

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3 Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle ABC = \angle ACD = \angle ADE$. Diagonals BD and CE meet at P. Prove that AP bisects side CD.

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4 Let $(F_n)_{n\geq o}$ be the sequence of Fibonacci numbers: $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for every $n \geq 0$.
Prove that for any prime $p \geq 3$, p divides $F_{2p} - F_p$.

– Day IV

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1 Prove that for any positive integer n there is an equiangular hexagon whose sidelengths are $n+1, n+2, ..., n+6$ in some order.

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2 Let $a_1, a_2, ..., a_n$ be real numbers such that $a_1 + a_2 + ... + a_n = 0$ and $|a_1| + |a_2| + ... + |a_n| = 1$. Prove that

$$|a_1 + 2a_2 + ... + na_n| \leq \frac{n-1}{2}$$

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3 Consider a triangle ABC. Let A_1 be the symmetric point of A with respect to the line BC, B_1 the symmetric point of B with respect to the line CA, and C_1 the symmetric point of C with respect to the line AB. Determine the possible set of angles of triangle ABC for which $A_1B_1C_1$ is equilateral.

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4 Let $p \geq 3$ be a prime. For $j = 1, 2, ..., p-1$, let r_j be the remainder when the integer $\frac{j^{p-1}-1}{p}$ is divided by p.
Prove that

$$r_1 + 2r_2 + ... + (p-1)r_{p-1} \equiv \frac{p+1}{2} (\mod p)$$