Greece Team Selection Test 2005

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1 The side lengths of a triangle are the roots of a cubic polynomial with rational coefficients. Prove that the altitudes of this triangle are roots of a polynomial of sixth degree with rational coefficients.

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2 Let Γ be a circle and let d be a line such that Γ and d have no common points. Further, let AB be a diameter of the circle Γ; assume that this diameter AB is perpendicular to the line d, and the point B is nearer to the line d than the point A. Let C be an arbitrary point on the circle Γ, different from the points A and B. Let D be the point of intersection of the lines AC and d. One of the two tangents from the point D to the circle Γ touches this circle Γ at a point E; hereby, we assume that the points B and E lie in the same halfplane with respect to the line AC. Denote by F the point of intersection of the lines BE and d. Let the line AF intersect the circle Γ at a point G, different from A.

Prove that the reflection of the point G in the line AB lies on the line CF.

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3 Let the polynomial $P(x) = x^3 + 19x^2 + 94x + a$ where $a \in \mathbb{N}$. If p a prime number, prove that no more than three numbers of the numbers $P(0), P(1), \ldots, P(p-1)$ are divisible by p.

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4 There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

i.) Each pair of students are in exactly one club.

ii.) For each student and each society, the student is in exactly one club of the society.

iii.) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is
in exactly m societies.

Find all possible values of k.

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