

Lotfi Zadeh Olympiad 2021, from Azerbaijan

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1 In the inscribed quadrilateral $ABCD$, P is the intersection point of diagonals and M is the midpoint of arc AB . Prove that line MP passes through the midpoint of segment CD , if and only if lines AB, CD are parallel.

2 Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be (not necessarily distinct) positive integers. We continue the sequences as follows: For every $i > n$, a_i is the smallest positive integer which is not among b_1, b_2, \dots, b_{i-1} , and b_i is the smallest positive integer which is not among a_1, a_2, \dots, a_{i-1} . Prove that there exists N such that for every $i > N$ we have $a_i = b_i$ or for every $i > N$ we have $a_{i+1} = a_i$.

3 Find the least possible value for the fraction

$$\frac{lcm(a, b) + lcm(b, c) + lcm(c, a)}{gcd(a, b) + gcd(b, c) + gcd(c, a)}$$

over all distinct positive integers a, b, c .

By $lcm(x, y)$ we mean the least common multiple of x, y and by $gcd(x, y)$ we mean the greatest common divisor of x, y .

4 Find the number of sequences of 0, 1 with length n satisfying both of the following properties:

- There exists a simple polygon such that its i -th angle is less than 180 degrees if and only if the i -th element of the sequence is 1.
- There exists a convex polygon such that its i -th angle is less than 90 degrees if and only if the i -th element of the sequence is 1.
