Dutch IMO Team Selection Test 2021

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–	Day 1	

1 The sequence of positive integers $a_0, a_1, a_2, ...$ is defined by $a_0 = 3$ and

$$a_{n+1} - a_n = n(a_n - 1)$$

for all $n \ge 0$. Determine all integers $m \ge 2$ for which $gcd(m, a_n) = 1$ for all $n \ge 0$.

2 Find all quadruplets (x_1, x_2, x_3, x_4) of real numbers such that the next six equalities apply:

$$\begin{cases} x_1 + x_2 = x_3^2 + x_4^2 + 6x_3x_4 \\ x_1 + x_3 = x_2^2 + x_4^2 + 6x_2x_4 \\ x_1 + x_4 = x_2^2 + x_3^2 + 6x_2x_3 \\ x_2 + x_3 = x_1^2 + x_4^2 + 6x_1x_4 \\ x_2 + x_4 = x_1^2 + x_3^2 + 6x_1x_3 \\ x_3 + x_4 = x_1^2 + x_2^2 + 6x_1x_2 \end{cases}$$

3 Let ABC be an acute-angled and non-isosceles triangle with orthocenter H. Let O be the center of the circumscribed circle of triangle ABC and let K be center of the circumscribed circle of triangle AHO. Prove that the reflection of K wrt OH lies on BC.

4 On a rectangular board with $m \times n$ squares ($m, n \ge 3$) there are dominoes (2×1 or 1×2 tiles), which do not overlap and do not extend beyond the board. Every domino covers exactly two squares of the board. Assume that the dominos cover the has the property that no more dominos can be added to the board and that the four corner spaces of the board are not all empty. Prove that at least $2/3$ of the squares of the board are covered with dominos.

– Day 2

1 Let Γ be the circumscribed circle of a triangle ABC and let D be a point at line segment BC. The circle passing through B and D tangent to Γ and the circle passing through $C$and D tangent to Γ intersect at a point $E \ne D$. The line DE intersects Γ at two points X and Y . Prove that $|EX| = |EY|$.

2 Stekel and Prick play a game on an $m \times n$ board, where m and n are positive are integers. They alternate turns, with Stekel starting. Spine bets on his turn, he always takes a pawn on a square where there is no pawn yet. Prick does his turn the same, but his pawn must always come into a square adjacent to the square that Spike just placed a pawn in on his previous turn. Prick wins like the whole board is full of pawns. Spike wins if Prik can no longer move a pawn on his turn, while there is still at least one empty square on the board. Determine for all pairs (m, n) who has a winning strategy.

3 Prove that for every positive integer n there are positive integers a and b exist with $n|4a^2+9b^2-1$.

4 Determine all positive integers n with the following property: for each triple (a, b, c) of positive real numbers there is a triple (k, ℓ, m) of non-negative integer numbers so that an^k, bn^ℓ and cn^m are the lengths of the sides of a (non-degenerate) triangle shapes.

– Day 3

1 Let m and n be natural numbers with mn even. Jetze is going to cover an $m \times n$ board (consisting of m rows and n columns) with dominoes, so that every domino covers exactly two squares, dominos do not protrude or overlap, and all squares are covered by a domino. Merlin then moves all the dominoe color red or blue on the board. Find the smallest non-negative integer V (in terms of m and n) so that Merlin can always ensure that in each row the number squares covered by a red domino and the number of squares covered by a blue one dominoes are not more than V, no matter how Jetze covers the board.

2 Let ABCbe a right triangle with $\angle C = 90^o$ and let D be the foot of the altitude from C. Let E be the centroid of triangle ACD and let F be the centroid of triangle BCD. The point P satisfies $\angle CEP = 90^o$ and $|CP| = |AP|$, while point Q satisfies $\angle CFQ = 90^o$ and $|CQ| = |BQ|$.
Prove that PQ passes through the centroid of triangle ABC.

3 Find all functions $f : R \to R$ with $f(x + yf(x + y)) = y^2 + f(x)f(y)$ for all $x, y \in R$.

4 Let $p > 10$ be prime. Prove that there are positive integers m and n with $m + n < p$ exist for which p is a divisor of $5^m7^n - 1$.