Berkeley Math Tournament , 2013 Spring, Analysis, Discrete, Geometry, Individual + Team Round

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–	Geometry Round
1	A rectangle with sides a and b has an area of 24 and a diagonal of length 11. Find the perimeter of this rectangle.
2	Two rays start from a common point and have an angle of 60 degrees. Circle C is drawn with radius 42 such that it is tangent to the two rays. Find the radius of the circle that has radius smaller than circle C and is also tangent to C and the two rays.
3	Given a regular tetrahedron $ABCD$ with center O, find $\sin \angle AOB$.
4	Two cubes A and B have different side lengths, such that the volume of cube A is numerically equal to the surface area of cube B. If the surface area of cube A is numerically equal to six times the side length of cube B, what is the ratio of the surface area of cube A to the volume of cube B?
5	Points A and B are fixed points in the plane such that $AB = 1$. Find the area of the region consisting of all points P such that $\angle APB > 120^o$
6	Let $ABCD$ be a cyclic quadrilateral where $AB = 4$, $BC = 11$, $CD = 8$, and $DA = 5$. If BC and DA intersect at X, find the area of $\triangle XAB$.
7	Let ABC be a triangle with $BC = 5, CA = 3$, and $AB = 4$. Variable points P, Q are on segments AB, AC, respectively such that the area of APQ is half of the area of ABC. Let x and y be the lengths of perpendiculars drawn from the midpoint of PQ to sides AB and AC, respectively. Find the range of values of $2y + 3x$.
8	ABC is an isosceles right triangle with right angle B and $AB = 1$. ABC has an incenter at E. The excircle to ABC at side AC is drawn and has center P. Let this excircle be tangent to AB at R. Draw T on the excircle so that RT is the diameter. Extend line BC and draw point D on BC so that DT is perpendicular to RT. Extend AC and let it intersect with DT at G. Let F be the incenter of CDG. Find the area of $\triangle EFP$.
9	Let ABC be a triangle. Points D, E, F are on segments BC, CA, AB, respectively. Suppose that $AF = 10$, $FB = 10$, $BD = 12$, $DC = 17$, $CE = 11$, and $EA = 10$. Suppose that the circumcircles of $\triangle BFD$ and $\triangle CED$ intersect again at X. Find the circumradius of $\triangle EXF$.

10 Let D, E, and F be the points at which the incircle, ω, of $\triangle\ ABC$ is tangent to BC, CA, and AB, respectively. AD intersects ω again at T. Extend rays TE, TF to hit line BC at E', F', respectively. If $BC = 21$, $CA = 16$, and $AB = 15$, then find $\left|\frac{1}{DE'} - \frac{1}{DF'}\right|$.

P1 Suppose a convex polygon has a perimeter of 1. Prove that it can be covered with a circle of radius $1/4$.

P2 From a point A construct tangents to a circle centered at point O, intersecting the circle at P and Q respectively. Let M be the midpoint of PQ. If K and L are points on circle O such that K, L, and A are collinear, prove $\angle MKO = \angle MLO$.

– Team Round

1 A time is called *reflexive* if its representation on an analog clock would still be permissible if the hour and minute hand were switched. In a given non-leap day ($12 : 00 : 00.00$ a.m. to $11 : 59 : 59.99$ p.m.), how many times are reflexive?

2 Find the sum of all positive integers N such that $s = \sqrt[3]{2 + \sqrt{N}} + \sqrt[3]{2 - \sqrt{N}}$ is also a positive integer

3 A round robin tennis tournament is played among 4 friends in which each player plays every other player only one time, resulting in either a win or a loss for each player. If overall placement is determined strictly by how many games each player won, how many possible placements are there at the end of the tournament? For example, Andy and Bob tying for first and Charlie and Derek tying for third would be one possible case.

4 Find the sum of all real numbers x such that $x^2 = 5x + 6\sqrt{x} - 3$.

5 Circle C_1 has center O and radius OA, and circle C_2 has diameter OA. AB is a chord of circle C_1 and BD may be constructed with D on OA such that BD and OA are perpendicular. Let C be the point where C_2 and BD intersect. If $AC = 1$, find AB.

6 In a class of 30 students, each students knows exactly six other students. (Of course, knowing is a mutual relation, so if A knows B, then B knows A). A group of three students is balanced if either all three students know each other, or no one knows anyone else within that group. How many balanced groups exist?

7 Consider the infinite polynomial $G(x) = F_1x + F_2x^2 + F_3x^3 + ...$ defined for $0 < x < \frac{\sqrt{5}-1}{2}$ where Fk is the kth term of the Fibonacci sequence defined to be $F_k = F_{k-1} + F_{k-2}$ with $F_1 = 1$, $F_2 = 1$. Determine the value a such that $G(a) = 2$.

8 A parabola has focus F and vertex V , where $VF = 10$. Let AB be a chord of length 100 that passes through F. Determine the area of $\triangle VAB$.

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9 Sequences x_n and y_n satisfy the simultaneous relationships $x_k = x_{k+1} + y_{k+1}$ and $x_k > y_k$ for all $k \geq 1$. Furthermore, either $y_k = y_{k+1}$ or $y_k = x_{k+1}$. If $x_1 = 3 + \sqrt{2}$, $x_3 = 5 - \sqrt{2}$, and $y_1 = y_5$, evaluate

$$(y_1)^2 + (y_2)^2 + (y_3)^2 + ...$$

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10 In a far away kingdom, there exist k^2 cities subdivided into k distinct districts, such that in the i^{th} district, there exist $2i - 1$ cities. Each city is connected to every city in its district but no cities outside of its district. In order to improve transportation, the king wants to add $k - 1$ roads such that all cities will become connected, but his advisors tell him there are many ways to do this. Two plans are different if one road is in one plan that is not in the other. Find the total number of possible plans in terms of k.

– Discrete Round

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1 A number is between 500 and 1000 and has a remainder of 6 when divided by 25 and a remainder of 7 when divided by 9. Find the only odd number to satisfy these requirements.

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2 If I roll three fair 4-sided dice, what is the probability that the sum of the resulting numbers is relatively prime to the product of the resulting numbers?

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3 Suppose we have 2013 piles of coins, with the ith pile containing exactly i coins. We wish to remove the coins in a series of steps. In each step, we are allowed to take away coins from as many piles as we wish, but we have to take the same number of coins from each pile. We cannot take away more coins than a pile actually has. What is the minimum number of steps we have to take?

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4 Given $f_1(x) = 2x - 2$ and, for $k \geq 2$, defined $f_k(x) = f(f_{k-1}(x))$ to be a real-valued function of x. Find the remainder when $f_{2013}(2012)$ is divided by the prime 2011.

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5 Consider the roots of the polynomial $x^{2013} - 2^{2013} = 0$. Some of these roots also satisfy $x^k - 2^k = 0$, for some integer $k < 2013$. What is the product of this subset of roots?

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6 A coin is flipped until there is a head followed by two tails. What is the probability that this will take exactly 12 flips?

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7 Denote by $S(a, b)$ the set of integers k that can be represented as $k = a \cdot m + b \cdot n$, for some non-negative integers m and n. So, for example, $S(2, 4) = \{0, 2, 4, 6, \ldots\}$. Then, find the sum of all possible positive integer values of x such that $S(18, 32)$ is a subset of $S(3, x)$.

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8 Let $f(n)$ take in a nonnegative integer n and return an integer between 0 and $n-1$ at random (with the exception being $f(0)=0$ always). What is the expected value of $f(f(22))$?

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9 2013 people sit in a circle, playing a ball game. When one player has a ball, he may only pass it to another player 3, 11, or 61 seats away (in either direction). If $f(A,B)$ represents the minimal number of passes it takes to get the ball from Person A to Person B, what is the maximal possible value of f?

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10 Let σ_n be a permutation of $\{1,\ldots,n\}$; that is, $\sigma_n(i)$ is a bijective function from $\{1,\ldots,n\}$ to itself. Define $f(\sigma)$ to be the number of times we need to apply σ to the identity in order to get the identity back. For example, f of the identity is just 1, and all other permutations have $f(\sigma)>1$. What is the smallest n such that there exists a σ_n with $f(\sigma_n)=k$?

P1 Ahuiliztli is playing around with some coins (pennies, nickels, dimes, and quarters). She keeps grabbing k coins and calculating the value of her handful. After a while, she begins to notice that if k is even, she more often gets even sums, and if k is odd, she more often gets odd sums. Help her prove this true! Given k coins chosen uniformly and at random, prove that. the probability that the parity of k is the same as the parity of the k coins' value is greater than the probability that the parities are different.

P2 Let p be an odd prime, and let $(p^p)! = mp^k$ for some positive integers m and k. Find in terms of p the number of ordered pairs (m,k) satisfying $m+k\equiv 0 \pmod p$.

– Analysis Round

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1 Find the value of a satisfying

$$\begin{aligned} a+b&=3\\ b+c&=11\\ c+a&=61 \end{aligned}$$

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2 A point P is given on the curve $x^4+y^4=1$. Find the maximum distance from the point P to the origin.

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3 Evaluate

$$\lim_{x\to 0}\frac{\sin 2x}{e^{3x}-e^{-3x}}$$

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4 Given a complex number z satisfies $\mathrm{Im}(z)=z^2-z$, find all possible values of $|z|$.

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5 Suppose that $c_n = (-1)^n(n+1)$. While the sum $\sum_{n=0}^{\infty} c_n$ is divergent, we can still attempt to assign a value to the sum using other methods. The Abel Summation of a sequence, a_n, is $\mathrm{Abel}(a_n) = \lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n$. Find $\mathrm{Abel}(c_n)$.

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6 The *minimal polynomial* of a complex number r is the unique polynomial with rational coefficients of minimal degree with leading coefficient 1 that has r as a root. If f is the minimal polynomial of $\cos\frac{\pi}{7}$, what is $f(-1)$?

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7 If x, y are positive real numbers satisfying $x^3 - xy + 1 = y^3$, find the minimum possible value of y.

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8 Billy is standing at $(1,0)$ in the coordinate plane as he watches his Aunt Sydney go for her morning jog starting at the origin. If Aunt Sydney runs into the First Quadrant at a constant speed of 1 meter per second along the graph of $x = \frac{2}{5}y^2$, find the rate, in radians per second, at which Billy's head is turning clockwise when Aunt Sydney passes through $x = 1$.

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9 Evaluate the integral

$$\int_0^1 \left(\sqrt{(x-1)^3+1} + x^{2/3} - (1-x)^{3/2} - \sqrt[3]{1-x^2}\right) dx$$

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10 Let the class of functions f_n be defined such that $f_1(x) = |x^3 - x^2|$ and $f_{k+1}(x) = |f_k(x) - x^3|$ for all $k \geq 1$. Denote by S_n the sum of all y-values of $f_n(x)$'s "sharp" points in the First Quadrant. (A "sharp" point is a point for which the derivative is not defined.) Find the ratio of odd to even terms,

$$\lim_{k\to\infty} \frac{S_{2k+1}}{S_{2k}}$$

P1 Prove that for all positive integers m and n,

$$\frac{1}{m}\cdot\binom{2n}{0} - \frac{1}{m+1}\cdot\binom{2n}{1} + \frac{1}{m+2}\cdot\binom{2n}{2} - \ldots + \frac{1}{m+2n}\cdot\binom{2n}{n2} > 0$$

P2 If $f(x) = x^n - 7x^{n-1} + 17x^{n-2} + a_{n-3}x^{n-3} + \ldots + a_0$ is a real-valued function of degree $n > 2$ with all real roots, prove that no root has value greater than 4 and at least one root has value less than 0 or greater than 2.

– Individual Round

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1 Billy the kid likes to play on escalators! Moving at a constant speed, he manages to climb up one escalator in 24 seconds and climb back down the same escalator in 40 seconds. If at any given moment the escalator contains 48 steps, how many steps can Billy climb in one second?

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2 S-Corporation designs its logo by linking together 4 semicircles along the diameter of a unit circle. Find the perimeter of the shaded portion of the logo.
`https://cdn.artofproblemsolving.com/attachments/8/6/f0eabd46f5f3a5806d49012b2f871a453b9e7`
`png`

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3 Two boxes contain some number of red, yellow, and blue balls. The first box has 3 red, 4 yellow, and 5 blue balls, and the second box has 6 red, 2 yellow, and 7 blue balls. There are two ways to select a ball from these boxes; one could first randomly choose a box and then randomly select a ball or one could put all the balls in the same box and simply randomly select a ball from there. How much greater is the probability of drawing a red ball using the second method than the first?

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4 Let $ABCD$ be a square with side length 2, and let a semicircle with flat side CD be drawn inside the square. Of the remaining area inside the square outside the semi-circle, the largest circle is drawn. What is the radius of this circle?

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5 Two positive integers m and n satisfy

$$max\left(m,n\right) = \left(m-n\right)^2$$

$$gcd\left(m,n\right) = \frac{min\left(m,n\right)}{6}$$

Find $lcm\left(m,n\right)$

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6 Bubble Boy and Bubble Girl live in bubbles of unit radii centered at $(20,13)$ and $(0,10)$ respectively. Because Bubble Boy loves Bubble Girl, he wants to reach her as quickly as possible, but he needs to bring a gift; luckily, there are plenty of gifts along the x-axis. Assuming that Bubble Girl remains stationary, find the length of the shortest path Bubble Boy can take to visit the x-axis and then reach Bubble Girl (the bubble is a solid boundary, and anything the bubble can touch, Bubble Boy can touch too)

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7 Given real numbers a,b,c such that $a+b-c = ab-bc-ca = abc = 8$. Find all possible values of a.

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8 The three-digit prime number p is written in base 2 as p_2 and in base 5 as p_5, and the two representations share the same last 2 digits. If the ratio of the number of digits in p_2 to the number of digits in p_5 is 5 to 2, find all possible values of p.

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9 An ant in the xy-plane is at the origin facing in the positive x-direction. The ant then begins a progression of moves, on the n^{th} of which it first walks $\frac{1}{5^n}$ units in the direction it is facing and then turns 60^o degrees to the left. After a very large number of moves, the ant's movements begins to converge to a certain point; what is the y-value of this point?

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10 If five squares of a 3×3 board initially colored white are chosen at random and blackened, what is the expected number of edges between two squares of the same color?

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11 Let $t = (a, b, c)$, and let us define $f^1(t) = (a+b, b+c, c+a)$ and $f^k(t) = f^{k-1}(f^1(t))$ for all $k > 1$. Furthermore, a permutation of t has the same elements, just in a different order (e.g., (b, c, a)). If $f^{2013}(s)$ is a permutation of s for some $s = (k, m, n)$, where k, m, and n are integers such that $|k|, |m|, |n| \leq 10$, how many possible values of s are there?

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12 Triangle ABC satisfies the property that $\angle A = a \log x$, $\angle B = a \log 2x$, and $\angle C = a \log 4x$ radians, for some real numbers a and x. If the altitude to side AB has length 8 and the altitude to side BC has length 9, find the area of $\triangle\ ABC$.

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13 Let $f(n)$ be a function from integers to integers. Suppose $f(11) = 1$, and $f(a)f(b) = f(a+b) + f(a-b)$, for all integers a, b. Find $f(2013)$.

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14 Triangle ABC has incircle O that is tangent to AC at D. Let M be the midpoint of AC. E lies on BC so that line AE is perpendicular to BO extended. If $AC = 2013$, $AB = 2014$, $DM = 249$, find CE.

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15 Let $ABCD$ be a convex quadrilateral with $\angle ABD = \angle BCD$, $AD = 1000$, $BD = 2000$, $BC = 2001$, and $DC = 1999$. Point E is chosen on segment DB such that $\angle ABD = \angle ECD$. Find AE.

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16 Find the sum of all possible n such that n is a positive integer and there exist a, b, c real numbers such that for every integer m, the quantity $\frac{2013m^3+am^2+bm+c}{n}$ is an integer.

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17 Let $N \geq 1$ be a positive integer and k be an integer such that $1 \leq k \leq N$. Define the recurrence $x_n = \frac{x_{n-1}+x_{n-2}+...+x_{n-N}}{N}$ for $n > N$ and $x_k = 1$, $x_1 = x_2 = ... = x_{k-1} = x_{k+1} = .. = x_N = 0$. As n approaches infinity, x_n approaches some value. What is this value?

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18 Paul and his pet octahedron like to play games together. For this game, the octahedron randomly draws an arrow on each of its faces pointing to one of its three edges. Paul then randomly chooses a face and progresses from face to adjacent face, as determined by the arrows on each face, and he wins if he reaches every face of the octahedron. What is the probability that Paul wins?

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19 Equilateral triangle ABC is inscribed in a circle. Chord AD meets BC at E. If $DE = 2013$, how many scenarios exist such that both DB and DC are integers (two scenarios are different if AB is different or AD is different)?

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20 A sequence a_n is defined such that $a_0 = \frac{1+\sqrt{3}}{2}$ and $a_{n+1} = \sqrt{a_n}$ for $n \geq 0$. Evaluate

$$\prod_{k=0}^{\infty} 1 - a_k + a_k^2$$