Berkeley Math Tournament , 2014 Spring, Analysis, Discrete, Geometry, Individual + Team Round

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–	Geometry Round	
1	Consider a regular hexagon with an incircle. What is the ratio of the area inside the incircle to the area of the hexagon?	
2	Regular hexagon $ABCDEF$ has side length 2 and center O. The point P is defined as the intersection of AC and OB. Find the area of quadrilateral $OPCD$.	
3	Consider an isosceles triangle ABC ($AB = BC$). Let D be on BC such that $AD \perp BC$ and O be a circle with diameter BC. Suppose that segment AD intersects circle O at E. If $CA = 2$ what is CE?	
4	A cylinder with length ℓ has a radius of 6 meters, and three spheres with radii $3, 4$, and 5 meters are placed inside the cylinder. If the spheres are packed into the cylinder such that ℓ is minimized, determine the length ℓ.	
5	In a 100-dimensional hypercube, each edge has length 1. The box contains $2^{100}+1$ hyperspheres with the same radius r. The center of one hypersphere is the center of the hypercube, and it touches all the other spheres. Each of the other hyperspheres is tangent to 100 faces of the hypercube. Thus, the hyperspheres are tightly packed in the hypercube. Find r.	
6	Square $ABCD$ has side length 5 and arc BD with center A. E is the midpoint of AB and CE intersects arc BD at F. G is placed onto BC such that FG is perpendicular to BC. What is the length of FG?	
7	Consider a parallelogram $ABCD$. E is a point on ray $\overrightarrow{AD}$. BE intersects AC at F and CD at G. If $BF = EG$ and $BC = 3$, find the length of AE	
8	Semicircle O has diameter $AB = 12$. Arc $AC = 135^o$. Let D be the midpoint of arc AC. Compute the region bounded by the lines CD and DB and the arc CB.	
9	Let ABC be a triangle. Construct points B' and C' such that ACB' and ABC' are equilateral triangles that have no overlap with $\triangle\ ABC$. Let BB' and CC' intersect at X. If $AX = 3$, $BC = 4$, and $CX = 5$, find the area of quadrilateral $BCB'C'$. .	

10 Consider 8 points that are a knight's move away from the origin (i.e., the eight points $\{(2,1)$, $(2,-1)$, $(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$, $(-2,1)$, $(-2,-1)\}$). Each point has probability $\frac{1}{2}$ of being visible. What is the expected value of the area of the polygon formed by points that are visible? (If exactly $0,1,2$ points appear, this area will be zero.)

P1 Let ABC be a triangle. Let r denote the inradius of $\triangle\ ABC$. Let r_a denote the A-exradius of $\triangle\ ABC$. Note that the A-excircle of $\triangle\ ABC$ is the circle that is tangent to segment BC, the extension of ray AB beyond B and the extension of AC beyond C. The A-exradius is the radius of the A-excircle. Define r_b and r_c analogously. Prove that

$$\frac{1}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}$$

P2 Let ABC be a fixed scalene triangle. Suppose that X,Y are variable points on segments AB, AC, respectively such that $BX=CY$. Prove that the circumcircle of $\triangle\ AXY$ passes through a fixed point other than A.

– Team Round

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1 What is the value of $1+7+21+35+35+21+7+1$?

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2 A mathematician is walking through a library with twenty-six shelves, one for each letter of the alphabet. As he walks, the mathematician will take at most one book off each shelf. He likes symmetry, so if the letter of a shelf has at least one line of symmetry (e.g., M works, L does not), he will pick a book with probability $\frac{1}{2}$. Otherwise he has a $\frac{1}{4}$ probability of taking a book. What is the expected number of books that the mathematician will take?

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3 Together, Abe and Bob have less than or equal to \$ 100. When Corey asks them how much money they have, Abe says that the reciprocal of his money added to Bob's money is thirteen times as much as the sum of Abe's money and the reciprocal of Bob's money. If Abe and Bob both have integer amounts of money, how many possible values are there for Abe's money?

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4 In a right triangle, the altitude from a vertex to the hypotenuse splits the hypotenuse into two segments of lengths a and b. If the right triangle has area T and is inscribed in a circle of area C, find ab in terms of T and C.

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5 Call two regular polygons supplementary if the sum of an internal angle from each polygon adds up to 180^o. For instance, two squares are supplementary because the sum of the internal angles is $90^o+90^o=180^o$. Find the other pair of supplementary polygons. Write your answer in the form (m,n) where m and n are the number of sides of the polygons and $m<n$.

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6 A train is going up a hill with vertical velocity given as a function of t by $\frac{1}{1-t^4}$, where t is between $[0,1)$. Determine its height as a function of t.

7 Let $VWXYZ$ be a square pyramid with vertex V with height 1, and with the unit square as its base. Let $STANFURD$ be a cube, such that face $FURD$ lies in the same plane as and shares the same center as square face $WXYZ$. Furthermore, all sides of $FURD$ are parallel to the sides of $WXYZ$. Cube $STANFURD$ has side length s such that the volume that lies inside the cube but outside the square pyramid is equal to the volume that lies inside the square pyramid but outside the cube. What is the value of s?

8 Annisa has n distinct textbooks, where $n > 6$. She has a different ways to pick a group of 4 books, b different ways to pick 5 books and c different ways to pick 6 books. If Annisa buys two more (distinct) textbooks, how many ways will she be able to pick a group of 6 books?

9 Two different functions f, g of x are selected from the set of real-valued functions

$$\left\{sinx, e^{-x}, x\ln x, \arctan x, \sqrt{x^2+x} - \sqrt{x^2+x} - x, \frac{1}{x}\right\}$$

to create a product function $f(x)g(x)$. For how many such products is $\lim_{x \to infty} f(x)g(x)$ finite?

10 A *unitary* divisor d of a number n is a divisor n that has the property $\gcd(d, n/d) = 1$. If $n = 1620$, what is the sum of all of the unitary divisors of d?

11 Suppose that $x^{10} + x + 1 = 0$ and $x^100 = a_0 + a_1x + ... + a_9x^9$. Find a_5.

12 A two-digit integer is *reversible* if, when written backwards in base 10, it has the same number of positive divisors. Find the number of reversible integers.

13 Let ABC be a triangle with $AB = 16$, $AC = 10$, $BC = 18$. Let D be a point on AB such that $4AD = AB$ and let E be the foot of the angle bisector from B onto AC. Let P be the intersection of CD and BE. Find the area of the quadrilateral $ADPE$.

14 Let (x, y) be an intersection of the equations $y = 4x^2 - 28x + 41$ and $x^2 + 25y^2 - 7x + 100y + \frac{349}{4} = 0$. Find the sum of all possible values of x.

15 Suppose a box contains 28 balls: 1 red, 2 blue, 3 yellow, 4 orange, 5 purple, 6 green, and 7 pink. One by one, each ball is removed uniformly at random and without replacement until all 28 balls have been removed. Determine the probability that the most likely "scenario of exhaustion" occurs; that is, determine the probability that the first color to have all such balls removed from the box is red, that the second is blue, the third is yellow, the fourth is orange, the fifth is purple, the sixth is green, and the seventh is pink.

16 Evaluate

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\min(n,k)\left(\frac{1}{2}\right)^n\left(\frac{1}{3}\right)^k$$

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17 Suppose you started at the origin on the number line in a coin-flipping game. Every time you flip a heads, you move forward one step, otherwise you move back one step. However, there are walls at positions 8 and -8; if you are at these positions and your coin flip dictates that you should move past them, instead you must stay. What is the expected number of coin flips needed to have visited both walls?

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18 Monty wants to play a game with you. He shows you five boxes, one of which contains a prize and four of which contain nothing. He allows you to choose one box but not to open it. He then opens one of the other four boxes that he knows to contain nothing. Then, he makes you switch and choose a different, unopened box. However, Monty sketchily reveals the contents of another (empty) box, selected uniformly at random from the two or three closed boxes (that you do not currently have chosen) that he knows to contain no prize. He then offers you the chance to switch again. Assuming you seek to maximize your return, determine the probability you get a prize.

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19 A number k is *nice* in base b if there exists a k-digit number n such that $n, 2n, ...kn$ are each some cyclic shifts of the digits of n in base b (for example, 2 is *nice* in base 5 because $2\cdot 135 = 315$). Determine all nice numbers in base 18.

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20 A certain type of Bessel function has the form $I(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}d\theta$ for all real x. Evaluate $\int_0^\infty xI(2x)e^{-x^2}dx$.

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2 If I roll three fair 4-sided dice, what is the probability that the sum of the resulting numbers is relatively prime to the product of the resulting numbers?

– Analysis Round

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1 Find all real numbers x such that $4^x - 2^{x+2} + 3 = 0$.

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2 Find the smallest positive value of x such that $x^3 - 9x^2 + 22x - 16 = 0$.

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3 Emma is seated on a train traveling at a speed of 120 miles per hour. She notices distance markers are placed evenly alongside the track, with a constant distance x between any two consecutive ones, and during a span of 6 minutes, she sees precisely 11 markers pass by her. Determine the difference (in miles) between the largest and smallest possible values of x.

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4 The function $f(x) = x^5 - 20x^4 + ax^3 + bx^2 + cx + 24$ has the interesting property that its roots can be arranged to form an arithmetic sequence. Determine $f(8)$.

5 Determine

$$\lim_{x\to\infty} \frac{\sqrt{x+2014}}{\sqrt{x}+\sqrt{x+2014}}$$

6 Find $f(2)$ given that f is a real-valued function that satisfies the equation

$$4f(x)+\left(\frac{2}{3}\right)(x^2+2)f\left(x-\frac{2}{x}\right)=x^3+1.$$

7 Let $f(x)=x^2+18$ have roots r_1 and r_2, and let $g(x)=x^2-8x+17$ have roots r_3 and r_4. If $h(x)=x^4+ax^3+bx^2+cx+d$ has roots r_1+r_3, r_1+r_4, r_2+r_3, and r_2+r_4, then find $h(4)$.

8 Suppose an integer-valued function f satisfies

$$\sum_{k=1}^{2n+1} f(k)=\ln|2n+1|-4\ln|2n-1| \text{ and } \sum_{k=0}^{2n} f(k)=4e^n-e^{n-1}$$

for all non-negative integers n. Determine $\sum_{n=0}^{\infty}\frac{f(n)}{2^n}$.

9 Find α such that

$$\lim_{x\to 0^+} x^\alpha I(x)=a \text{ given } I(x)=\int_0^\infty \sqrt{1+t}\cdot e^{-xt}dt$$

where a is a nonzero real number.

10 Suppose that $x^3-x+10^{-6}=0$. Suppose that $x_1<x_2<x_3$ are the solutions for x. Find the integers (a,b,c) closest to 10^8x_1, 10^8x_2, and 10^8x_3 respectively.

P1 Suppose that a,b,c,d are non-negative real numbers such that $a^2+b^2+c^2+d^2=2$ and $ab+bc+cd+da=1$. Find the maximum value of $a+b+c+d$ and determine all equality cases.

P2 Define $\eta(f)$ to be the number of roots that are repeated of the complex-valued polynomial f (e.g. $\eta((x-1)^3\cdot(x+1)^4)=5$). Prove that for nonconstant, relatively prime $f,g\in\mathbb{C}[x]$,

$$\eta(f)+\eta(g)+\eta(f+g)<\deg f+\deg g$$

– Discrete Round

1 For the team, power, and tournament rounds, BMT divided up the teams into 14 rooms. You sign up to proctor all 3 rounds, but you cannot proctor in the same room more than once.
How many ways can you be assigned for rooms for the 3 rounds?

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2 Find the number of 5-digit n, s.t. every digit of n is either $0, 1, 3$, or 4, and n is divisible by 15.

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3 The Professor chooses to assign homework problems from a set of problems labeled 1 to 100, inclusive. He will not assign two problems whose numbers share a common factor greater than 1. If the Professor chooses to assign the maximum number of homework problems possible, how many different combinations of problems can he assign?

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4 What is the sum of the first 31 integers that can be written as a sum of distinct powers of 3?

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5 Alice, Bob, and Chris each roll 4 dice. Each only knows the result of their own roll. Alice claims that there are at least 5 multiples of 3 among the dice rolled. Bob has 1 six and no threes, and knows that Alice wouldn't claim such a thing unless he had at least 2 multiples of 3. Bob can call Alice a liar, or claim that there are at least 6 multiples of 3, but Chris says that he will immediately call Bob a liar if he makes this claim. Bob wins if he calls Alice a liar and there aren't at least 5 multiples of 3, or if he claims there are at least 6 multiples of 3, and there are. What is the probability that Bob loses no matter what he does?

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6 Pick a 3-digit number abc, which contains no 0's. The probability that this is a winning number is $\frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c}$. However, the BMT problem writer tries to balance out the chances for the numbers in the following ways:

- For the lowest digit n in the number, he rolls an n-sided die for each time that the digit appears, and gives the number 0 probability of winning if an n is rolled.
- For the largest digit m in the number, he rolls an m-sided die once and scales the probability of winning by that die roll.

If you choose optimally, what is the probability that your number is a winning number?

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7 For a positive integer n, let $\phi(n)$ denote the number of positive integers between 1 and n, inclusive, which are relatively prime to n. We say that a positive integer k is total if $k = \frac{n}{\phi(n)}$, for some positive integer n. Find all total numbers.

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8 Suppose that positive integers $a_1, a_2, \ldots, a_{2014}$ (not necessarily distinct) satisfy the condition that: $\frac{a_1}{a_2}, \frac{a_2}{a_3}, \ldots, \frac{a_{2013}}{a_{2014}}$ are pairwise distinct. What is the minimal possible number of distinct numbers in $\{a_1, a_2, \ldots, a_{2014}\}$?

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9 Leo and Paul are at the Berkeley BART station and are racing to San Francisco. Leo is planning to take the line that takes him directly to SF, and because he has terrible BART luck, his train will arrive in some integer number of minutes, with probability $\frac{i}{210}$ for $1 \leq i \leq 20$ at any given minute. Paul will take a second line, whose trains always arrive before Leo's train, with uniform probability. However, Paul must also make a transfer to a 3rd line, whose trains arrive with uniform probability between 0 and 10 minutes after Paul reaches the transfer station. What is the probability that Leo gets to SF before Paul does?

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10 Let f be a function on $(1, \dots, n)$ that generates a permutation of $(1, \dots, n)$. We call a fixed point of f any element in the original permutation such that the element's position is not changed when the permutation is applied. Given that n is a multiple of 4, g is a permutation whose fixed points are $(1, \dots, \frac{n}{2})$, and h is a permutation whose fixed points consist of every element in an even-numbered position. What is the expected number of fixed points in $h(g(1, 2, \dots, 104))$?

P1 Let a simple polygon be defined as a polygon in which no consecutive sides are parallel and no two non-consecutive sides share a common point. Given that all vertices of a simple polygon P are lattice points (in a Cartesian coordinate system, each vertex has integer coordinates), and each side of P has integer length, prove that the perimeter must be even.

P2 Given an integer $n \geq 2$, the graph G is defined by:

- Vertices of G are represented by binary strings of length n
- Two vertices a, b are connected by an edge if and only if they differ in exactly 2 places

Let S be a subset of the vertices of G, and let S' be the set of edges between vertices in S and vertices not in S. Show that if $|S|$ (the size of S) $\leq 2^{n-2}$, then $|S'| \geq |S|$.

– Individual Round

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1 A *festive* number is a four-digit integer containing one of each of the digits $0, 1, 2$, and 4 in its decimal representation. How many festive numbers are there?

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2 Suppose $\triangle ABC$ is similar to $\triangle DEF$, with A, B, and C corresponding to D, E, and F respectively. If $\overline{AB} = \overline{EF}$, $\overline{BC} = \overline{FD}$, and $\overline{CA} = \overline{DE} = 2$, determine the area of $\triangle ABC$.

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3 Suppose three boba drinks and four burgers cost 28 dollars, while two boba drinks and six burgers cost $\$37.70$. If you paid for one boba drink using only pennies, nickels, dimes, and quarters, determine the least number of coins you could use.

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4 Alice, Bob, Cindy, David, and Emily sit in a circle. Alice refuses to sit to the right of Bob, and Emily sits next to Cindy. If David sits next to two girls, determine who could sit immediately to the right of Alice.

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5 Fred and George are playing a game, in which Fred flips 2014 coins and George flips 2015 coins. Fred wins if he flips at least as many heads as George does, and George wins if he flips more heads than Fred does. Determine the probability that Fred wins.

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6 Let m and n be integers such that $m + n$ and $m - n$ are prime numbers less than 100. Find the maximal possible value of mn.

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7 If $f(x, y) = 3x^2 + 3xy + 1$ and $f(a, b) + 1 = f(b, a) = 42$, then determine $|a + b|$.

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8 Line segment AB has length 4 and midpoint M. Let circle C_1 have diameter AB, and let circle C_2 have diameter AM. Suppose a tangent of circle C_2 goes through point B to intersect circle C_1 at N. Determine the area of triangle AMN.

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9 Suppose $a_1, a_2, ...$ and $b_1, b_2, ...$ are sequences satisfying $a_n + b_n = 7$, $a_n = 2b_{n-1} - a_{n-1}$, and $b_n = 2a_{n-1} - b_{n-1}$, for all n. If $a_1 = 2$, find $(a_{2014})^2 - (b_{2014})^2$.
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10 A plane intersects a sphere of radius 10 such that the distance from the center of the sphere to the plane is 9. The plane moves toward the center of the bubble at such a rate that the increase in the area of the intersection of the plane and sphere is constant, and it stops once it reaches the center of the circle. Determine the distance from the center of the sphere to the plane after two-thirds of the time has passed.

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11 Suppose x, y, and 1 are side lengths of a triangle$T$ such that $x < 1$ and $y < 1$. Given x and y are chosen uniformly at random from all possible pairs (x, y), determine the probability that T is obtuse.

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12 Suppose four coplanar points A, B, C, and D satisfy $AB = 3$, $BC = 4$, $CA = 5$, and $BD = 6$. Determine the maximal possible area of $\triangle ACD$.

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13 A cylinder is inscribed within a sphere of radius 10 such that its volume is *almost-half* that of the sphere. If *almost-half* is defined such that the cylinder has volume $\frac{1}{2} + \frac{1}{250}$ times the sphere's volume, find the sum of all possible heights for the cylinder.

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14 Suppose that $f(x) = \frac{x}{x^2-2x+2}$ and $g(x_1, x_2, ..., x_7) = f(x_1) + f(x_2) + ... + f(x_7)$. If $x_1, x_2, ..., x_7$ are non-negative real numbers with sum 5, determine for how many tuples $(x_1, x_2, ..., x_7)$ does $g(x_1, x_2, ..., x_7)$ obtain its maximal value.

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15 Albert and Kevin are playing a game. Kevin has a 10% chance of winning any given round in the match. If Kevin wins the first game, he wins the match. If not, he requests that the match be extended to a best of 3. If he wins the best of 3, he wins the match. If not, then he requests the match be extended to a best of 5, and so forth. What is the probability that Kevin eventually wins the match? (A best of $2n+1$ match consists of a series of rounds. The first person to reach $n+1$ winning games wins the match)

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16 Let n be the smallest positive integer such that the number obtained by taking n's rightmost digit (decimal expansion) and moving it to be the leftmost digit is 7 times n. Determine the number of digits in n.

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17 A convex solid is formed in four-dimensional Euclidean space with vertices at the 24 possible permutations of $\{1,2,3,4\}$ (so $(1,2,3,4)$, $(1,2,4,3)$, etc.). What is the product of the number of faces and edges of this solid?

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18 Suppose the polynomial $f(x) = x^{2014}$ is equal to $f(x) = \sum_{k=0}^{2014} a_k \binom{x}{k}$ for some real numbers $a_0, ..., a_{2014}$. Find the largest integer m such that 2^m divides a_{2013}.

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19 Evaluate the integral $\int_0^{\pi/2} \sqrt{\tan\theta} d\theta$.

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20 Suppose three circles of radius 5 intersect at a common point. If the three (other) pairwise intersections between the circles form a triangle of area 8, find the radius of the smallest possible circle containing all three circles.
