Berkeley Math Tournament , 2015 Spring, Geometry, Individual + Team Round

www.artofproblemsolving.com/community/c2746594

by parmenides51, jasperE3

–	Geometry Round
1	Let ABC be a triangle. The angle bisectors of $\angle ABC$ and $\angle ACB$ intersect at D. If $\angle BAC = 80^o$, what are all possible values for $\angle BDC$?
2	$ABCDEF$ is a regular hexagon. Let R be the overlap between $\vartriangle ACE$ and $\vartriangle BDF$. What is the area of R divided by the area of $ABCDEF$?
3	Let M be on segmentBC of $\vartriangle ABC$ so that $AM = 3$, $BM = 4$, and $CM = 5$. Find the largest possible area of $\vartriangle ABC$.
4	Let $ABCD$ be a rectangle. Circles C_1 and C_2 are externally tangent to each other. Furthermore, C_1 is tangent to AB and AD, and C_2 is tangent to CB and CD. If $AB = 18$ and $BC = 25$, then find the sum of the radii of the circles.
5	Let $A = (1, 0)$, $B = (0, 1)$, and $C = (0, 0)$. There are three distinct points, P, Q, R, such that $\{A, B, C, P\}$, $\{A, B, C, Q\}$, $\{A, B, C, R\}$ are all parallelograms (vertices unordered). Find the area of $\vartriangle PQR$.
6	Let C be the sphere $x^2 + y^2 + (z - 1)^2 = 1$. Point P on C is $(0, 0, 2)$. Let $Q = (14, 5, 0)$. If PQ intersects C again at Q', then find the length PQ' .
7	Define $A = (1, 0, 0)$, $B = (0, 1, 0)$, and P as the set of all points (x, y, z) such that $x + y + z = 0$. Let P be the point on P such that $d = AP + PB$ is minimized. Find d^2.
8	Suppose that $A = (\frac12, \sqrt3)$. Suppose that B, C, D are chosen on the ellipse $x^2 + (y/2)^2 = 1$ such that the area of $ABCD$ is maximized. Assume that A, B, C, D lie on the ellipse going counterclockwise. What are all possible values of B?
9	Let ABC be a triangle. Suppose that a circle with diameter BC intersects segments CA, AB at E, F, respectively. Let D be the midpoint of BC. Suppose that AD intersects EF at X. If $AB = \sqrt9$, $AC = \sqrt{10}$, and $BC = \sqrt{11}$, what is $\frac{EX}{XF}$?
10	Let ABC be a triangle with points E, F on CA, AB, respectively. Circle C_1 passes through E, F and is tangent to segment BC at D. Suppose that $AE = AF = EF = 3$, $BF = 1$, and $CE = 2$. What is $\frac{ED^2}{FD^2}$?

P1 Suppose that circles C_1 and C_2 intersect at X and Y. Let A, B be on C_1, C_2, respectively, such that A, X, B lie on a line in that order. Let A, C be on C_1, C_2, respectively, such that A, Y, C lie on a line in that order. Let A', B', C' be another similarly defined triangle with $A \neq A'$. Prove that $BB' = CC'$.

P2 Suppose that fixed circle C_1 with radius $a > 0$ is tangent to the fixed line ℓ at A. Variable circle C_2, with center X, is externally tangent to C_1 at $B \neq A$ and ℓ at C. Prove that the set of all X is a parabola minus a point

Tie 1 Let $ABCD$ be a parallelogram. Suppose that E is on line DC such that C lies on segment ED. Then say lines AE and BD intersect at X and lines CX intersects AB at F. If $AB = 7$,$BC = 13$, and $CE = 91$, then find $\frac{AF}{FB}$.

Tie 2 The unit square $ABCD$ has E as midpoint of AD and a circle of radius r tangent to AB, BC, and CE. Determine r.

Tie 3 The permutohedron of order 3 is the hexagon determined by points $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, and $(3, 2, 1)$. The pyramid determined by these six points and the origin has a unique inscribed sphere of maximal volume. Determine its radius.

– Team Round

1 A fair 6-sided die is repeatedly rolled until a $1, 4, 5$, or 6 is rolled. What is the expected value of the product of all the rolls?

2 Compute the sum of the digits of 1001^{10}

3 How many ways are there to place the numbers $2, 3, ..., 10$ in a 3×3 grid, such that any two numbers that share an edge are mutually prime?

4 Triangle ABC has side lengths $AB = 3$, $BC = 4$, and $CD = 5$. Draw line ℓ_A such that ℓ_A is parallel to BC and splits the triangle into two polygons of equal area. Define lines ℓ_B and ℓ_C analogously. The intersection points of ℓ_A, ℓ_B, and ℓ_C form a triangle. Determine its area.

5 Determine the smallest positive integer containing only 0 and 1 as digits that is divisible by each integer 1 through 9.

6 Consider the set $S = \{1, 2, ..., 2015\}$. How many ways are there to choose 2015 distinct (possibly empty and possibly full) subsets $X_1, X_2, ..., X_{2015}$ of S such that X_i is strictly contained in X_{i+1} for all $1 \leq i \leq 2014$?

7 $X_1, X_2, ..., X_{2015}$ are 2015 points in the plane such that for all $1 \leq i, j \leq 2015$, the line segment $X_iX_{i+1} = X_jX_{j+1}$ and angle $\angle X_iX_{i+1}X_{i+2} = \angle X_jX_{j+1}X_{j+2}$ (with cyclic indices such that $X_{2016} = X_1$ and $X_{2017} = X_2$). Given fixed X_1 and X_2, determine the number of possible locations for X_3.

---

8 The sequence $(x_n)_{n \in N}$ satisfies $x_1 = 2015$ and $x_{n+1} = \sqrt[3]{13x_n - 18}$ for all $n \geq 1$. Determine $\lim_{n \to \infty} x_n$.

---

9 Find the side length of the largest square that can be inscribed in the unit cube.

---

10 Quadratics $g(x) = ax^2 + bx + c$ and $h(x) = dx^2 + ex + f$ are such that the six roots of g, h, and $g - h$ are distinct real numbers (in particular, they are not double roots) forming an arithmetic progression in some order. Determine all possible values of a/d.

---

11 Write down $1, 2, 3, ..., 2015$ in a row on a whiteboard. Every minute, select a pair of adjacent numbers at random, erase them, and insert their sum where you selected the numbers. (For instance, selecting 3 and 4 from $1, 2, 3, 4, 5$ would result in $1, 2, 7, 5$.) Repeat this process until you have two numbers remaining. What is the probability that the smaller number is less than or equal to 2015?

---

12 Let $f(n)$ be the number of ordered pairs (k, ℓ) of positive integers such that $n = (2\ell - 1) \cdot 2^k - k$, and let $g(n)$ be the number of ordered pairs (k, ℓ) of positive integers such that $n = \ell \cdot 2^{k+1} - k$. Compute $\sum_{i=1}^{\infty} \frac{f(i) - g(i)}{2^i}$.

.

---

13 There exist right triangles with integer side lengths such that the legs differ by 1. For example, $3 - 4 - 5$ and $20 - 21 - 29$ are two such right triangles. What is the perimeter of the next smallest Pythagorean right triangle with legs differing by 1?

---

14 Alice is at coordinate point $(0, 0)$ and wants to go to point $(11, 6)$. Similarly, Bob is at coordinate point $(5, 6)$ and wants to go to point $(16, 0)$. Both of them choose a lattice path from their current position to their target position at random (such that each lattice path has an equal probability of being chosen), where a lattice path is defined to be a path composed of unit segments with orthogonal direction (parallel to x-axis or y-axis) and of minimal length. (For instance, there are six lattice paths from $(0, 0)$ to $(2, 2)$.) If they walk with the same speed, find the probability that they meet.

---

15 Compute

$$\int_{1/2}^{2} \frac{x^2 + 1}{x^2(x^{2015} + 1)} dx.$$

---

16 Five points A, B, C, D, and E in three-dimensional Euclidean space have the property that $AB = BC = CD = DE = EA = 1$ and $\angle ABC = \angle BCD = \angle CDE = \angle DEA = 90^o$. Find all possible $\cos(\angle EAB)$.

---

17 There exist real numbers x and y such that $x(a^3 + b^3 + c^3) + 3yabc \geq (x + y)(a^2b + b^2c + c^2a)$ holds for all positive real numbers a, b, and c. Determine the smallest possible value of x/y.
.

---

18 A value $x \in [0, 1]$ is selected uniformly at random. A point (a, b) is called *friendly* to x if there exists a circle between the lines $y = 0$ and $y = 1$ that contains both (a, b) and $(0, x)$. Find the area of the region of the plane determined by possible locations of friendly points.

---

19 Two sequences $(x_n)_{n\in N}$ and $(y_n)_{n\in N}$ are defined recursively as follows:

$x_0 = 2015$ and $x_{n+1} = \left\lfloor x_n \cdot \frac{y_{n+1}}{y_{n-1}} \right\rfloor$ for all $n \geq 0$,

$y_0 = 307$ and $y_{n+1} = y_n + 1$ for all $n \geq 0$.

Compute $\lim_{n\to\infty} \frac{x_n}{(y_n)^2}$.

---

20 The Tower of Hanoi is a puzzle with n disks of different sizes and 3 vertical rods on it. All of the disks are initially placed on the leftmost rod, sorted by size such that the largest disk is on the bottom. On each turn, one may move the topmost disk of any nonempty rod onto any other rod, provided that it is smaller than the current topmost disk of that rod, if it exists. (For instance, if there were two disks on different rods, the smaller disk could move to either of the other two rods, but the larger disk could only move to the empty rod.) The puzzle is solved when all of the disks are moved to the rightmost rod. The specifications normally include an intelligent monk to move the disks, but instead there is a monkey making random moves (with each valid move having an equal probability of being selected). Given 64 disks, what is the expected number of moves the monkey will have to make to solve the puzzle?

– Analysis Round

---

1 Let x, y, z, w be integers such that $2^x + 2^y + 2^z + 2^w = 24.375$. Find the value of $xyzw$.

---

2 Let $g(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots$. Find the coefficient of x^{2015} of $f(x) = \frac{g(x)}{1-x}$.

---

3 Find all integer solutions to

$$x^2 + 2y^2 + 3z^2 = 36,$$
$$3x^2 + 2y^2 + z^2 = 84,$$
$$xy + xz + yz = -7.$$

4 Let $\{a_n\}$ be a sequence of real numbers with $a_1 = -1$, $a_2 = 2$ and for all $n \geq 3$,

$$a_{n+1} - a_n - a_{n+2} = 0.$$

Find $a_1 + a_2 + a_3 + \ldots + a_{2015}$.

5 Let x and y be real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. If the maximum and minimum values of $x^2 + y^2$ are M and m respectively, compute the numerical value of $M - m$.

6 The roots of the equation $x^5 - 180x^4 + Ax^3 + Bx^2 + Cx + D = 0$ are in geometric progression. The sum of their reciprocals is 20. Compute $|D|$.

7 Evaluate $\sum_{k=0}^{37}(-1)^k\binom{75}{2k}$.

8 Let ω be a primitive 7th root of unity. Find

$$\prod_{k=0}^{6}\left(1 + \omega^k - \omega^{2k}\right).$$

(A complex number is a primitive root of unity if and only if it can be written in the form $e^{2k\pi i/n}$, where k is relatively prime to n.)

9 Find

$$\lim_{n\to\infty}\frac{1}{n^3}\left(\sqrt{n^2-1^2} + \sqrt{n^2-2^2} + \ldots + \sqrt{n^2-(n-1)^2}\right).$$

10 Evaluate

$$\int_0^{\pi/2}\ln(4\sin x)dx.$$

P1 Suppose $z_0, z_1, \ldots, z_{n-1}$ are complex numbers such that $z_k = e^{2k\pi i/n}$ for $k = 0, 1, 2, \ldots, n-1$. Prove that for any complex number z, $\sum_{k=0}^{n-1}|z - z_k| \geq n$.

P2 Let $f(x)$ be a nonconstant monic polynomial of degree n with rational coefficents that is irreducible, meaning it cannot be factored into two nonconstant rational polynomials. Find and prove a formula for the number of monic complex polynomials that divide f.

– Discrete Round

1 Alice is planning a trip from the Bay Area to one of 5 possible destinations (each of which is serviced by only 1 airport) and wants to book two flights, one to her destination and one returning. There are 3 airports within the Bay Area from which she may leave and to which she may return. In how many ways may she plan her flight itinerary?

---

2 Determine the largest integer n such that 2^n divides the decimal representation given by some permutation of the digits 2, 0, 1, and 5. (For example, 2^1 divides 2150. It may start with 0.)

---

3 How many rational solutions are there to $5x^2 + 2y^2 = 1$?

---

4 Determine the greatest integer N such that N is a divisor of $n^{13} - n$ for all integers n.

---

5 Three balloon vendors each offer two types of balloons - one offers red & blue, one offers blue & yellow, and one offers yellow & red. I like each vendor the same, so I must buy 7 balloons from each. How many different possible triples (x, y, z) are there such that I could buy x blue, y yellow, and z red balloons?

---

6 There are 30 cities in the empire of Euleria. Every week, Martingale City runs a very well-known lottery. 900 visitors decide to take a trip around the empire, visiting a different city each week in some random order. 3 of these cities are inhabited by mathematicians, who will talk to all visitors about the laws of statistics. A visitor with this knowledge has probability 0 of buying a lottery ticket, else they have probability 0.5 of buying one. What is the expected number of visitors who will play the Martingale Lottery?

---

7 At Durant University, an A grade corresponds to raw scores between 90 and 100, and a B grade corresponds to raw scores between 80 and 90. Travis has 3 equally weighted exams in his math class. Given that Travis earned an A on his first exam and a B on his second (but doesn't know his raw score for either), what is the minimum score he needs to have a 90% chance of getting an A in the class? Note that scores on exams do not necessarily have to be integers.

---

8 Two players play a game with a pile of N coins on a table. On a player's turn, if there are n coins, the player can take at most $n/2+1$ coins, and must take at least one coin. The player who grabs the last coin wins. For how many values of N between 1 and 100 (inclusive) does the first player have a winning strategy?

---

9 There exists a unique pair of positive integers k, n such that k is divisible by 6, and $\sum_{i=1}^{k} i^2 = n^2$. Find (k, n).

---

10 A partition of a positive integer n is a summing $n_1 + \ldots + n_k = n$, where $n_1 \geq n_2 \geq \ldots \geq n_k$. Call a partition *perfect* if every $m \leq n$ can be represented uniquely as a sum of some subset of the n_i's. How many perfect partitions are there of $n = 307$?

P1 Find two disjoint sets N_1 and N_2 with $N_1 \cup N_2 = \mathbb{N}$, so that neither set contains an infinite arithmetic progression.

P2 Suppose $k > 3$ is a divisor of $2^p + 1$, where p is prime. Prove that $k \geq 2p + 1$.

– Individual Round

---

1 The boba shop sells four different types of milk tea, and William likes to get tea each weekday. If William refuses to have the same type of tea on successive days, how many different combinations could he get, Monday through Friday?

---

2 Suppose we list the decimal representations of the positive even numbers from left to right. Determine the 2015^{th} digit in the list.

---

3 A quadrilateral $ABCD$ has a right angle at $\angle ABC$ and satisfies $AB = 12$, $BC = 9$, $CD = 20$, and $DA = 25$. Determine BD^2.
.

---

4 A train traveling at 80 mph begins to cross a 1 mile long bridge. At this moment, a man begins to walk from the front of the train to the back of the train at a speed of 5 mph. The man reaches the back of the train as soon as the train is completely off the bridge. What is the length of the train (as a fraction of a mile)?

---

5 Find the number of ways to partition a set of 10 elements, $S = \{1, 2, 3, ..., 10\}$ into two parts; that is, the number of unordered pairs $\{P, Q\}$ such that $P \cup Q = S$ and $P \cap Q = \emptyset$.

---

6 An integer-valued function f satisfies $f(2) = 4$ and $f(mn) = f(m)f(n)$ for all integers m and n. If f is an increasing function, determine $f(2015)$.

---

7 In $\triangle\ ABC$, $\angle B = 46^o$ and $\angle C = 48^o$. A circle is inscribed in $\triangle\ ABC$ and the points of tangency are connected to form PQR. What is the measure of the largest angle in $\triangle\ PQR$?

---

8 An integer is between 0 and 999999 (inclusive) is chosen, and the digits of its decimal representation are summed. What is the probability that the sum will be 19?

---

9 The number 2^{29} has a 9-digit decimal representation that contains all but one of the 10 (decimal) digits. Determine which digit is missing

---

10 We have 10 boxes of different sizes, each one big enough to contain all the smaller boxes when put side by side. We may nest the boxes however we want (and how deeply we want), as long as we put smaller boxes in larger ones. At the end, all boxes should be directly or indirectly nested in the largest box. How many ways can we nest the boxes?

---

11 Let r, s, and t be the three roots of the equation $8x^3 + 1001x + 2008 = 0$. Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

---

12 How many possible arrangements of bishops are there on a 8×8 chessboard such that no bishop threatens a square on which another lies and the maximum number of bishops are used? (Note that a bishop threatens any square along a diagonal containing its square.)

---

13 On a 2×40 chessboard colored black and white in the standard alternating pattern, 20 rooks are placed randomly on the black squares. The expected number of white squares with only rooks as neighbors can be expressed as a/b, where a and b are coprime positive integers. What is $a + b$? (Two squares are said to be neighbors if they share an edge.)

---

14 Determine

$$\left|\prod_{k=1}^{10}(e^{\frac{i\pi}{2^k}} + 1)\right|$$

---

15 Recall that an icosahedron is a 3-dimensional solid characterized by its 20 congruent faces, each of which is an equilateral triangle. Determine the number of rigid rotations that preserve the symmetry of the icosahedron. (Each vertex moves to the location of another vertex.)

---

16 A binary decision tree is a list of n yes/no questions, together with instructions for the order in which they should be asked (without repetition). For instance, if $n = 3$, there are 12 possible binary decision trees, one of which asks question 2 first, then question 3 (followed by question 1) if the answer was yes or question 1 (followed by question 3) if the answer was no. Determine the greatest possible k such that 2^k divides the number of binary decision trees on $n = 13$ questions.

---

17 A circle intersects square $ABCD$ at points A, E, and F, where E lies on AB and F lies on AD, such that $AE + AF = 2(BE + DF)$. If the square and the circle each have area 1, determine the area of the union of the circle and square.

---

18 Evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(3n-1)}$.

---

19 It is known that 4 people A, B, C, and D each have a $1/3$ probability of telling the truth. Suppose that • A makes a statement. • B makes a statement about the truthfulness of A's statement. • C makes a statement about the truthfulness of B's statement. • D says that C says that B says that A was telling the truth.
What is the probability that A was actually telling the truth?

---

20 Let a and b be real numbers for which the equation $2x^4 + 2ax^3 + bx^2 + 2ax + 2 = 0$ has at least one real solution. For all such pairs (a, b), find the minimum value of $8a^2 + b^2$.

Tie 1 Compute the surface area of a rectangular prism with side lengths $2, 3, 4$.

Tie 2 Let $S_n = 1 + 2 +, , , +n$. Define

$$T_n = \frac{S_2}{S_2 - 1} \cdot \frac{S_3}{S_3 - 1} \cdot ... \cdot \frac{S_n}{S_n - 1}.$$

Find T_{2015}.

Tie 3 A bag contains 12 marbles: 3 red, 4 green, and 5 blue. Repeatedly draw marbles with replacement until you draw two marbles of the same color in a row. What is the expected number of times that you will draw a marble?