Berkeley Math Tournament , 2017 Spring, Geometry, Discrete, Analysis, Individual + Team Round

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– Geometry Round

1 What is the largest n such that there exists a non-degenerate convex n-gon such that each of its angles are an integer number of degrees, and are all distinct?

2 Let S be the set of points A in the xy-plane such that the four points A, $(2,3)$, $(-1,0)$, and $(0,6)$ form the vertices of a parallelogram. Let P be the convex polygon whose vertices are the points in S. What is the area of P?

3 Let $ABCDEF$ be a regular hexagon with side length 1. Now, construct square $AGDQ$. What is the area of the region inside the hexagon and not the square?

4 How many lattice points (v,w,x,y,z) does a 5-sphere centered on the origin, with radius 3, contain on its surface or in its interior?

5 Suppose the side lengths of triangle ABC are the roots of polynomial $x^3 - 27x^2 + 222x - 540$. What is the product of its inradius and circumradius?

6 Given a cube with side length 1, we perform six cuts as follows: one cut parallel to the xy-plane, two cuts parallel to the yz-plane, and three cuts parallel to the xz-plane, where the cuts are made uniformly independent of each other. What is the expected value of the volume of the largest piece?

7 Determine the maximal area triangle such that all of its vertices satisfy $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

8 Given a circle of radius 25, consider the set of triangles with area at least 768. What is the area of the intersection of all the triangles in this set?

9 Let $\triangle ABC$ be a triangle. Let D be the point on BC such that DA is tangent to the circumcircle of ABC. Let E be the point on the circumcircle of ABC such that DE is tangent to the circumcircle of ABC, but $E \neq A$. Let F be the intersection of AE and BC. Given that $BF/FC = 4/5$, find the maximum possible value for $\sin \angle ACB$/

10 Colorado and Wyoming are both defined to be 4 degrees tall in latitude and 7 degree wide in longitude. In particular, Colorado is defined to be at 37^oN to 41^oN, and $102^o03'W$ to $109^o03'W$, whereas Wyoming is defined to be 41^oN to 45^oN, and $104^o03'W$ to $111^o03'W$. Assuming Earth is a perfect sphere with radius R, what is the ratio of the areas of Wyoming to Colorado, in terms of R?

– Team Round

1 You are racing an Artificially Intelligent Robot, called AI, that you built. You can run at a constant speed of 10 m/s throughout the race. Meanwhile, AI starts running at a constant speed of 1 m/s. Thereafter, when exactly 1 second has passed from when AI last changed its speed, AI's speed instantaneously becomes 1 m/s faster, so that AI runs at a constant speed of k m/s in the kth second of the race. (Start counting seconds at 1). Suppose AI beats you by exactly 1 second. How many meters was the race?

2 Colin has 900 Choco Pies. He realizes that for some integer values of $n \leq 900$, if he eats n pies a day, he will be able to eat the same number of pies every day until he runs out. How many possible values of n are there?

3 Suppose we have $w < x < y < z$, and each of the 6 pairwise sums are distinct. The 4 greatest sums are $4, 3, 2, 1$. What is the sum of all possible values of w?

4 2 darts are thrown randomly at a circular board with center O, such that each dart has an equal probability of hitting any point on the board. The points at which they land are marked A and B. What is the probability that $\angle AOB$ is acute?

5 You enter an elevator on floor 0 of a building with some other people, and request to go to floor 10. In order to be efficient, it doesn't stop at adjacent floors (so, if it's at floor 0, its next stop cannot be floor 1). Given that the elevator will stop at floor 10, no matter what other floors it stops at, how many combinations of stops are there for the elevator?

6 The center of a square of side length 1 is placed uniformly at random inside a circle of radius 1. Given that we are allowed to rotate the square about its center, what is the probability that the entire square is contained within the circle for some orientation of the square?

7 There are 86400 seconds in a day, which can be deduced from the conversions between seconds, minutes, hours, and days. However, the leading scientists decide that we should decide on 3 new integers x, y, and z, such that there are x seconds in a minute, y minutes in an hour, and z hours in a day, such that $xyz = 86400$ as before, but such that the sum $x + y + z$ is minimized. What is the smallest possible value of that sum?

8 A function f with its domain on the positive integers $N = \{1, 2, ...\}$ satisfies the following conditions:
(a) $f(1) = 2017$.
(b) $\sum_{i=1}^{n} f(i) = n^2 f(n)$, for every positive integer $n > 1$.
What is the value of $f(2017)$?

9 Let $AB = 10$ be a diameter of circle P. Pick point C on the circle such that $AC = 8$. Let the circle with center O be the incircle of $\triangle ABC$. Extend line AO to intersect circle P again at D. Find the length of BD.

10 You and your friend play a game on a 7×7 grid of buckets. Your friend chooses 5 "lucky" buckets by marking an "X" on the bottom that you cannot see. However, he tells you that they either form a vertical, or horizontal line of length 5. To clarify, he will select either of the following sets of buckets:
either $\{(a, b), (a, b + 1), (a, b + 2), (a, b + 3), (a, b + 4)\}$,
or $\{(b, a), (b + 1, a), (b + 2, a), (b + 3, a), (b + 4, a)\}$,
with $1 \le a \le 7$, and $1 \le b \le 3$. Your friend lets you pick up at most n buckets, and you win if one of the buckets you picked was a "lucky" bucket. What is the minimum possible value of n such that, if you pick your buckets optimally, you can guarantee that at least one is "lucky"?

11 Ben picks a positive number n less than 2017 uniformly at random. Then Rex, starting with the number 1, repeatedly multiplies his number by n and then finds the remainder when dividing by 2017. Rex does this until he gets back to the number 1. What is the probability that, during this process, Rex reaches every positive number less than 2017 before returning back to 1?

12 A robot starts at the origin of the Cartesian plane. At each of 10 steps, he decides to move 1 unit in any of the following directions: left, right, up, or down, each with equal probability. After 10 steps, the probability that the robot is at the origin is $\frac{n}{4^{10}}$. Find$n$

13 4 equilateral triangles of side length 1 are drawn on the interior of a unit square, each one of which shares a side with one of the 4 sides of the unit square. What is the common area enclosed by all 4 equilateral triangles?

14 Suppose that there is a set of 2016 positive numbers, such that both their sum, and the sum of their reciprocals, are equal to 2017. Let x be one of those numbers. Find the maximum possible value of $x + \frac{1}{x}$.
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15 In triangle ABC, the angle at C is 30^o, side BC has length 4, and side AC has length 5. Let P be the point such that triangle ABP is equilateral and non-overlapping with triangle ABC. Find the distance from C to P.

– Individual Round

1 In three years, Xingyou's age in years will be twice his current height in feet. If Xingyou's current age in years is also his current height in feet, what is Xingyou's age in years right now?

2 Barack is an equilateral triangle and Michelle is a square. If Barack and Michelle each have perimeter 12, find the area of the polygon with larger area.

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3 How many letters in the word UNCOPYRIGHTABLE have at least one line of symmetry?

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4 There are two 3-digit numbers which end in 99. These two numbers are also the product of two integers which differ by 2. What is the sum of these two numbers?

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5 How many pairs of positive integers (a, b) satisfy the equation $log_a 16 = b$?

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6 For how many numbers n does 2017 divided by n have a remainder of either 1 or 2?

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7 What is the sum of the infinite series $\frac{20}{3} + \frac{17}{9} + \frac{20}{27} + \frac{17}{81} + \frac{20}{243} + \frac{17}{729} + ...$?

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8 If $xy = 15$ and $x + y = 11$, calculate the value of $x^3 + y^3$.

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9 The digits $1, 4, 9$ and 2 are each used exactly once to form some 4-digit number N. What is the sum of all possible values of N?

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10 Let S be the set of points A in the Cartesian plane such that the four points A, $(2, 3)$, $(-1, 0)$, and $(0, 6)$ form the vertices of a parallelogram. Let P be the convex polygon whose vertices are the points in S. What is the area of P?

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11 Naomi has a class of 100 students who will compete with each other in five teams. Once the teams are made, each student will shake hands with every other student, except the students in his or her own team. Naomi chooses to partition the students into teams so as to maximize the number of handshakes. How many handshakes will there be?

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12 Square S is the unit square with vertices at $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. We choose a random point (x, y) inside S and construct a rectangle with length x and width y. What is the average of $\lfloor p \rfloor$ where p is the perimeter of the rectangle? $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

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13 Two points are located 10 units apart, and a circle is drawn with radius r centered at one of the points. A tangent line to the circle is drawn from the other point. What value of r maximizes the area of the triangle formed by the two points and the point of tangency?

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14 Let x be the first term in the sequence $31, 331, 3331, ...$ which is divisible by 17. How many digits long is$x$?

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15 Alice and Bob live on the edges and vertices of the unit cube. Alice begins at point $(0, 0, 0)$ and Bob begins at $(1, 1, 1)$. Every second, each of them chooses one of the three adjacent corners and walks at a constant rate of 1 unit per second along the edge until they reach the corner, after which they repeat the process. What is the expected amount of time in seconds before Alice and Bob meet?

16 Let ABC be a triangle with $AB = 3$, $BC = 5$, $AC = 7$, and let P be a point in its interior. If G_A, G_B, G_C are the centroids of $\triangle\ PBC$, $\triangle\ PAC$, $\triangle\ PAB$, respectively, find the maximum possible area of $\triangle\ G_AG_BG_C$.

17 Triangle ABC is drawn such that $\angle A = 80^o$, $\angle B = 60^o$, and $\angle C = 40^o$. Let the circumcenter of $\triangle\ ABC$ be O, and let ω be the circle with diameter AO. Circle ω intersects side AC at point P. Let M be the midpoint of side BC, and let the intersection of ω and PM be K. Find the measure of $\angle MOK$.

18 Consider the sequence (k_n) defined by $k_{n+1} = n(k_n + k_{n-1})$ and $k_0 = 0$, $k_1 = 1$. What is $\lim_{n\to\infty} \frac{k_n}{n!}$?

19 Let T be the triangle in the xy-plane with vertices $(0, 0)$, $(3, 0)$, and $\left(0, \frac{3}{2}\right)$. Let E be the ellipse inscribed in T which meets each side of T at its midpoint. Find the distance from the center of E to $(0, 0)$.

20 Evaluate $\sum_{k=0}^{15} \left(2^{560}(-1)^k \cos^{560}\left(\frac{k\pi}{16}\right)\right) \pmod{17}$.

– Discrete Round

1 You have 9 colors of socks and 5 socks of each type of color. Pick two socks randomly. What is the probability that they are the same color?

2 Each BMT, every student chooses one of three focus rounds to take. Bob plans to attend BMT for the next 4 years and wants to gure out what focus round to take each year. Given that he wants to take each focus round at least once, how many ways can he choose which round to take each year?

3 What is the smallest positive integer with exactly 7 distinct proper divisors?

4 What is the greatest multiple of 9 that can be formed by using each of the digits in the set $\{1, 3, 5, 7, 9\}$ at most once.

5 How many subsets of $\{1, 2, ..., 9\}$ do not contain 2 adjacent numbers?

6 Let $S = \{1, 2, ..., 6\}$. How many functions $f : S \to S$ are there such that for all $s \in S$,

$$f^5(s) = f(f(f(f(f(s))))) = 1?$$

7 A light has been placed on every lattice point (point with integer coordinates) on the (infi nite) $2D$ plane. De ne the Chebyshev distance between points (x_1, y_1) and (x_2, y_2) to be $max(|x_1 - x_2|, |y_1 - y_2|)$. Each light is turned on with probability $\frac{1}{2^{d/2}}$, where d is the Chebyshev distance from that point to the origin. What is expected number of lights that have all their directly adjacent lights turned on? (Adjacent points being points such that $|x_1 - x_2| + |y_1 - y_2| = 1$.)

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8 In a 1024 person randomly seeded single elimination tournament bracket, each player has a unique skill rating. In any given match, the player with the higher rating has a $\frac{3}{4}$ chance of winning the match. What is the probability the second lowest rated player wins the tournament?

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9 n balls are placed independently uniformly at random into n boxes. One box is selected at random, and is found to contain b balls. Let e_n be the expected value of b^4. Find

$$\lim_{n\to\infty} e_n.$$

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10 Let $\phi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Evaluate

$$\sum_{n=1}^{64} (-1)^n \left\lfloor \frac{64}{n} \right\rfloor \phi(n).$$

– Analysis Round

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1 10 students take the Analysis Round. The average score was a 3 and the high score was a 7. If no one got a 0, what is the maximum number of students that could have achieved the high score?

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2 Find all solutions to $3^x - 9^{x-1} = 2$.

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3 Compute $\int_{-9}^{9} 17x^3 \cos(x^2) dx$.

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4 Find the value of $\frac{1}{2} + \frac{4}{2^2} + \frac{9}{2^3} + \frac{16}{2^4} + ...$

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5 Find the value of y such that the following equation has exactly three solutions.

$$||x - 1| - 4| = y.$$

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6 Consider the function $f(x, y, z) = (x - y + z, y - z + x, z - x + y)$ and denote by $f^{(n)}(x, y, z)$ the function f applied n times to the tuple (x, y, z). Let r_1, r_2, r_3 be the three roots of the equation

$x^3 - 4x^2 + 12 = 0$ and let $g(x) = x^3 + a_2x^2 + a_1x + a_0$ be the cubic polynomial with the tuple $f^{(3)}(r_1, r_2, r_3)$ as roots. Find the value of a_1.

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7 Compute

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)(2k+1)}$$

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8 The numerical value of the following integral

$$\int_0^1 (-x^2 + x)^{2017} \lfloor 2017x \rfloor dx$$

can be expressed in the form $a\frac{m!^2}{n!}$ where a is minimized. Find $a + m + n$.
(Note $\lfloor x \rfloor$ is the largest integer less than or equal to x.)

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9 Let a_d be the number of non-negative integer solutions (a, b) to $a + b = d$ where $a \equiv b$ (mod n) for a fixed $n \in Z^+$. Consider the generating function $M(t) = a_0 + a_1t + a_2t^2 + ...$ Consider

$$P(n) = \lim_{t \to 1} \left(nM(t) - \frac{1}{(1-t)^2} \right).$$

Then $P(n), n \in Z^+$ is a polynomial in n, so we can extend its domain to include all real numbers while having it remain a polynomial. Find $P(0)$.

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10 De ne $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Evaluate $\sum_{n=1}^{2017} \binom{2017}{n} H_n (-1)^n$.
