Greece Team Selection Test 2008

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1 Find all possible values of $a \in \mathbb{R}$ and $n \in \mathbb{N}^*$ such that $f(x) = (x-1)^n + (x-2)^{2n+1} + (1-x^2)^{2n+1} + a$

is divisible by $\phi(x) = x^2 - x + 1$

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2 In a village X_0 there are 80 tourists who are about to visit 5 nearby villages X_1, X_2, X_3, X_4, X_5.Each of them has chosen to visit only one of them.However,there are cases when the visit in a village forces the visitor to visit other villages among X_1, X_2, X_3, X_4, X_5.Each tourist visits only the village he has chosen and the villages he is forced to.If X_1, X_2, X_3, X_4, X_5 are totally visited by $40, 60, 65, 70, 75$ tourists respectively,then find how many tourists had chosen each one of them and determine all the ordered pairs $(X_i, X_j) : i, j \in \{1, 2, 3, 4, 5\}$ which are such that,the visit in X_i forces the visitor to visit X_j as well.

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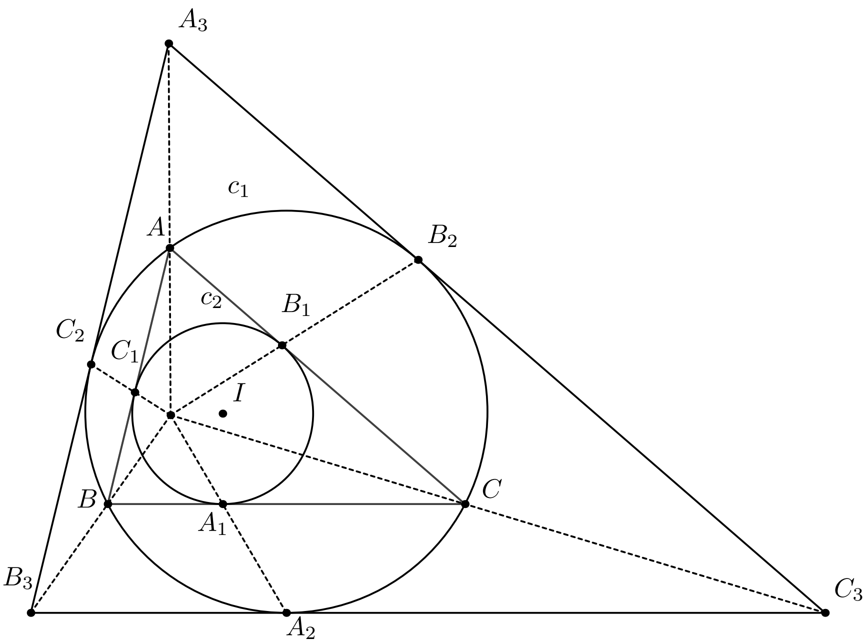
2 The bisectors of the angles $\angle A, \angle B, \angle C$ of a triangle $\triangle ABC$ intersect with the circumcircle $c_1(O, R)$ of $\triangle ABC$ at A_2, B_2, C_2 respectively.The tangents of c_1 at A_2, B_2, C_2 intersect each other at A_3, B_3, C_3 (the points A_3, A lie on the same side of BC,the points B_3, B on the same side of CA,and C_3, C on the same side of AB).The incircle $c_2(I, r)$ of $\triangle ABC$ is tangent to BC, CA, AB at A_1, B_1, C_1 respectively.Prove that $A_1A_2, B_1B_2, C_1C_2, AA_3, BB_3, CC_3$ are concurrent.



4 Given is the equation $x^2 + y^2 - axy + 2 = 0$ where a is a positive integral parameter.

i.Show that,for $a \neq 4$ there exist no pairs (x, y) of positive integers satisfying the equation.

ii. Show that,for $a = 4$ there exist infinite pairs (x, y) of positive integers satisfying the equation,and determine those pairs.