Harvard-MIT Mathematics Tournament 2021

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– Algebra and Number Theory

1 Compute the sum of all positive integers n for which the expression

$$\frac{n+7}{\sqrt{n-1}}$$

is an integer.

2 Compute the number of ordered pairs of integers (a, b), with $2 \leq a, b \leq 2021,$ that satisfy the equation

$$a^{\log_b\left(a^{-4}\right)} = b^{\log_a\left(ba^{-3}\right)}.$$

3 Among all polynomials $P(x)$ with integer coefficients for which $P(-10) = 145$ and $P(9) = 164$, compute the smallest possible value of $|P(0)|$.

4 Suppose that $P(x, y, z)$ is a homogeneous degree 4 polynomial in three variables such that $P(a, b, c) = P(b, c, a)$ and $P(a, a, b) = 0$ for all real a, b, and c. If $P(1, 2, 3) = 1$, compute $P(2, 4, 8)$.

Note: $P(x, y, z)$ is a homogeneous degree 4 polynomial if it satisfies $P(ka, kb, kc) = k^4P(a, b, c)$ for all real k, a, b, c.

5 Let n be the product of the first 10 primes, and let

$$S = \sum_{xy|n} \varphi(x) \cdot y,$$

where $\varphi(x)$ denotes the number of positive integers less than or equal to x that are relatively prime to x, and the sum is taken over ordered pairs (x, y) of positive integers for which xy divides n. Compute $\frac{S}{n}$.

6 Suppose that m and n are positive integers with $m < n$ such that the interval $[m, n)$ contains more multiples of 2021 than multiples of 2000. Compute the maximum possible value of $n - m$.

7 Suppose that x, y, and z are complex numbers of equal magnitude that satisfy

$$x + y + z = -\frac{\sqrt{3}}{2} - i\sqrt{5}$$

and

$$xyz = \sqrt{3} + i\sqrt{5}.$$

If $x = x_1 + ix_2, y = y_1 + iy_2$, and $z = z_1 + iz_2$ for real x_1, x_2, y_1, y_2, z_1 and z_2 then

$$(x_1x_2 + y_1y_2 + z_1z_2)^2$$

can be written as $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

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8 For positive integers a and b, let $M(a, b) = \frac{\text{lcm}(a,b)}{\text{gcd}(a,b)}$, and for each positive integer $n \geq 2$, define

$$x_n = M(1, M(2, M(3, \ldots, M(n-2, M(n-1, n)) \cdots))).$$

Compute the number of positive integers n such that $2 \leq n \leq 2021$ and $5x_n^2 + 5x_{n+1}^2 = 26x_nx_{n+1}$.

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9 Let f be a monic cubic polynomial satisfying $f(x) + f(-x) = 0$ for all real numbers x. For all real numbers y, define $g(y)$ to be the number of distinct real solutions x to the equation $f(f(x)) = y$. Suppose that the set of possible values of $g(y)$ over all real numbers y is exactly $\{1, 5, 9\}$. Compute the sum of all possible values of $f(10)$.

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10 Let S be a set of positive integers satisfying the following two conditions:
• For each positive integer n, at least one of $n, 2n, \ldots, 100n$ is in S.
• If a_1, a_2, b_1, b_2 are positive integers such that $\gcd(a_1a_2, b_1b_2) = 1$ and $a_1b_1, a_2b_2 \in S$, then $a_2b_1, a_1b_2 \in S$.
Suppose that S has natural density r. Compute the minimum possible value of $\lfloor 10^5 r \rfloor$.
Note: S has natural density r if $\frac{1}{n}|S \cap 1, \ldots, n|$ approaches r as n approaches ∞.

– Combinatorics

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1 Leo the fox has a 5 by 5 checkerboard grid with alternating red and black squares. He fills in the grid with the numbers $1, 2, 3, \ldots, 25$ such that any two consecutive numbers are in adjacent squares
(sharing a side) and each number is used exactly once. He then computes the sum of the numbers in
the 13 squares that are the same color as the center square. Compute the maximum possible sum Leo
can obtain.

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2 Ava and Tiffany participate in a knockout tournament consisting of a total of 32 players. In each of 5 rounds, the remaining players are paired uniformly at random. In each pair, both players are equally likely to win, and the loser is knocked out of the tournament. The probability that Ava and Tiffany play each other during the tournament is $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

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3 Let N be a positive integer. Brothers Michael and Kylo each select a positive integer less than or equal to N, independently and uniformly at random. Let p_N denote the probability that the product of these two integers has a units digit of 0. The maximum possible value of p_N over all possible choices of N can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

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4 Let $S = \{1, 2, \ldots, 9\}$. Compute the number of functions $f : S \to S$ such that, for all $s \in S, f(f(f(s))) = s$ and $f(s) - s$ is not divisible by 3.

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5 Teresa the bunny has a fair 8-sided die. Seven of its sides have fixed labels $1, 2, \cdots, 7$, and the label on the eighth side can be changed and begins as 1. She rolls it several times, until each of $1, 2, \ldots, 7$ appears at least once. After each roll, if k is the smallest positive integer that she has not rolled so far, she relabels the eighth side with k. The probability that 7 is the last number she rolls is $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

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6 A light pulse starts at a corner of a reflective square. It bounces around inside the square, reflecting off of the square's perimeter n times before ending in a different corner. The path of the light pulse, when traced, divides the square into exactly 2021 regions. Compute the smallest possible value of n.

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7 Let $S = \{1, 2, \ldots, 2021\}$, and let $\mathcal{F}$ denote the set of functions $f : S \to S$. For a function $f \in \mathcal{F}$, let

$$T_f = \{f^{2021}(s) : s \in S\},$$

where $f^{2021}(s)$ denotes $f(f(\cdots(f(s))\cdots))$ with 2021 copies of f. Compute the remainder when

$$\sum_{f \in \mathcal{F}} |T_f|$$

is divided by the prime 2017, where the sum is over all functions f in $\mathcal{F}$.

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8 Compute the number of ways to fill each cell in a 8×8 square grid with one of the letters H, M, or T such that every 2×2 square in the grid contains the letters H, M, M, T in some order.

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9 An up-right path between two lattice points P and Q is a path from P to Q that takes steps of length 1 unit either up or to the right.

How many up-right paths from $(0, 0)$ to $(7, 7)$, when drawn in the plane with the line $y = x - 2.021$, enclose exactly one bounded region below that line?

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10 Jude repeatedly flips a coin. If he has already flipped n heads, the coin lands heads with probability $\frac{1}{n+2}$ and tails with probability $\frac{n+1}{n+2}$. If Jude continues flipping forever, let p be the probability that he flips 3 heads in a row at some point. Compute $\lfloor 180p \rfloor$.

– Geometry

1 A circle contains the points $(0, 11)$ and $(0, -11)$ on its circumference and contains all points (x, y) with $x^2 + y^2 < 1$ in its interior. Compute the largest possible radius of the circle.

2 Let X_0 be the interior of a triangle with side lengths $3, 4,$ and 5. For all positive integers n, define X_n to be the set of points within 1 unit of some point in X_{n-1}. The area of the region outside X_{20} but inside X_{21} can be written as $a\pi + b$, for integers a and b. Compute $100a + b$.

3 Triangle ABC has a right angle at C, and D is the foot of the altitude from C to AB. Points $L, M,$ and N are the midpoints of segments $AD, DC,$ and CA, respectively. If $CL = 7$ and $BM = 12$, compute BN^2.

4 Let ABCD be a trapezoid with $AB \parallel CD, AB = 5, BC = 9, CD = 10,$ and $DA = 7$. Lines BC and DA intersect at point E. Let M be the midpoint of CD, and let N be the intersection of the circumcircles of $\triangle BMC$ and $\triangle DMA$ (other than M). If $EN^2 = \frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

5 Let AEF be a triangle with $EF = 20$ and $AE = AF = 21$. Let B and D be points chosen on segments AE and AF, respectively, such that BD is parallel to EF. Point C is chosen in the interior of triangle AEF such that $ABCD$ is cyclic. If $BC = 3$ and $CD = 4$, then the ratio of areas $\frac{[ABCD]}{[AEF]}$ can be written as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

6 In triangle ABC, let M be the midpoint of BC, H be the orthocenter, and O be the circumcenter. Let N be the reflection of M over H. Suppose that $OA = ON = 11$ and $OH = 7$. Compute BC^2.

7 Let O and A be two points in the plane with $OA = 30$, and let Γ be a circle with center O and radius r. Suppose that there exist two points B and C on Γ with $\angle ABC = 90^\circ$ and $AB = BC$. Compute the minimum possible value of $\lfloor r \rfloor$.

8 Two circles with radii 71 and 100 are externally tangent. Compute the largest possible area of a right triangle whose vertices are each on at least one of the circles.

9 Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $AD = BD$. Let M be the midpoint of AB, and let $P \neq C$ be the second intersection of the circumcircle of $\triangle BCD$ and the diagonal AC. Suppose that $BC = 27, CD = 25,$ and $AP = 10$. If $MP = \frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

10 Acute triangle ABC has circumcircle Γ. Let M be the midpoint of BC. Points P and Q lie on Γ so that $\angle APM = 90^\circ$ and $Q \neq A$ lies on line AM. Segments PQ and BC intersect at S. Suppose that $BS = 1, CS = 3, PQ = 8\sqrt{\frac{7}{37}}$, and the radius of Γ is r. If the sum of all possible values of

r^2 can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a+b$.

– Team

1 Let a and b be positive integers with $a>b$. Suppose that

$$\sqrt{\sqrt{a}+\sqrt{b}}+\sqrt{\sqrt{a}-\sqrt{b}}$$

is an integer.
(a) Must $\sqrt{a}$ be an integer?
(b) Must $\sqrt{b}$ be an integer?

2 Let ABC be a right triangle with $\angle A=90^\circ$. A circle ω centered on BC is tangent to AB at D and AC at E. Let F and G be the intersections of ω and BC so that F lies between B and G. If lines DG and EF intersect at X, show that $AX=AD$.

3 Let m be a positive integer. Show that there exists a positive integer n such that each of the $2m+1$ integers

$$2^n-m, 2^n-(m-1), \dots, 2^n+(m-1), 2^n+m$$

is positive and composite.

4 Let k and n be positive integers and let

$$S=\{(a_1,\dots,a_k)\in\mathbb{Z}^k \mid 0\le a_k\le\cdots\le a_1\le n, a_1+\cdots+a_k=n\}$$

.
Determine, with proof, the value of

$$\sum_{(a_1,\dots,a_k)\in S}\binom{n}{a_1}\binom{a_1}{a_2}\cdots\binom{a_{k-1}}{a_k}$$

in terms of k and n, where the sum is over all k-tuples in S.

5 A convex polyhedron has n faces that are all congruent triangles with angles $36^\circ, 72^\circ$, and 72°. Determine, with proof, the maximum possible value of n.

6 Let $f(x)=x^2+x+1$. Determine, with proof, all positive integers n such that $f(k)$ divides $f(n)$ whenever k is a positive divisor of n.

7 In triangle ABC, let M be the midpoint of BC and D be a point on segment AM. Distinct points Y and Z are chosen on rays $\overrightarrow{CA}$ and $\overrightarrow{BA}$, respectively, such that $\angle DYC=\angle DCB$ and $\angle DBC=\angle DZB$. Prove that the circumcircle of ΔDYZ is tangent to the circumcircle of ΔDBC.

8 For each positive real number α, define

$$\lfloor \alpha\mathbb{N} \rfloor := \{\lfloor \alpha m \rfloor \mid m \in \mathbb{N}\}.$$

Let n be a positive integer. A set $S \subseteq \{1, 2, \ldots, n\}$ has the property that: for each real $\beta > 0$,

$$\text{if } S \subseteq \lfloor \beta\mathbb{N} \rfloor, \text{then } \{1, 2, \ldots, n\} \subseteq \lfloor \beta\mathbb{N} \rfloor.$$

Determine, with proof, the smallest positive size of S.

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9 Let scalene triangle ABC have circumcenter O and incenter I. Its incircle ω is tangent to sides $BC, CA,$ and AB at $D, E,$ and F, respectively. Let P be the foot of the altitude from D to EF, and let line DP intersect ω again at $Q \neq D$. The line OI intersects the altitude from A toBC at T. Given that $OI \| BC$, show that $PQ = PT$.

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10 Let $n > 1$ be a positive integer. Each unit square in an $n \times n$ grid of squares is colored either black or white,
such that the following conditions hold:

• Any two black squares can be connected by a sequence of black squares where every two consecutive squares in the sequence share an edge; • Any two white squares can be connected by a sequence of white squares where every two consecutive squares in the sequence share an edge; • Any 2×2 subgrid contains at least one square of each color.

Determine, with proof, the maximum possible difference between the number of black squares and white squares in this grid (in terms of n).
