Final Round of 2021 Durer Math Competition

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by parmenides51

–   Category E

–   Day 1

1   Show that if the difference of two positive cube numbers is a positive prime, then this prime number has remainder 1 after division by 6.

2   In the country of Óxisz the minister of finance observed at the end of the tax census that the sum of properties of any two neighboring city counted in dinar is divisible by 1000, and she also observed that the sum of properties of all cities is also divisible by 1000. What is the least sum of properties of all cities if the map of the cities looks as follows?
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png
Remark: The cities may have non-integer properties, but it is also positive. On the map the points are the cities, and two cities are neighboring if there is a direct connection between them.

3   Let A and B different points of a circle k centered at O in such a way such that AB is not a diagonal of k. Furthermore, let X be an arbitrary inner point of the segment AB. Let k_1 be the circle that passes through the points A and X, and A is the only common point of k and k_1. Similarly, let k_2 be the circle that passes through the points B and X, and B is the only common point of k and k_2. Let M be the second intersection point of k_1 and k_2. Let Q denote the center of circumscribed circle of the triangle AOB. Let O_1 and O_2 be the centers of k_1 and k_2. Show that the points M, O, O_1, O_2, Q are on a circle.

4   Indians find those sequences of non-negative real numbers $x_0, x_1, ...$ *mystical* t hat satisfy $x_0 < 2021$, $x_{i+1} = \lfloor x_i \rfloor \{x_i\}$ for every $i \geq 0$, furthermore the sequence contains an integer different from 0. How many sequences are mystical according to the Indians?

5   A torpedo set consists of 2 pieces of 1×4, 4 pieces of 1×3, 6 pieces of 1×2 and 8 pieces of 1×1 ships.
a) Can one put the whole set to a 10×10 table so that the ships do not even touch with corners? (The ships can be placed both horizontally and vertically.)
b) Can we solve this problem if we change 4 pieces of 1×1 ships to 3 pieces of 1×2 ships?
c) Can we solve the problem if we change the remaining 4 pieces of 1×1 ships to one piece of 1×3 ship and one piece of 1×2 ship? (So the number of pieces are $2, 5, 10, 0$.)

6   (Game) In an Indian reservatory there are 15 totem poles arranged according to the left figure.

Silent Stream and Red Fire used to play the following game: In turns they stretch ropes between two-two poles in such a way that every stretched rope is parallel to a side of the big triangle and no rope can go along a pole that is already touched by another rope. Furthermore, if instead of a rope one can stretch out a straight line extension of the rope, then one should stretch out this extension. The one who cannot stretch out more rope according to the rules loses.

Win two games in a row against the organizers! You can decide that you want to start or to be the second player. The figure on the right depicts the first three steps of a game. First Silent Stream stretches the blue rope, then Red Fire stretches the red one, finally Silent Stream stretches the blue one. `https://cdn.artofproblemsolving.com/attachments/f/8/3b8b9e38a8a477da288566ecb26036bfc`
`png`

–   Day 2

1   In Sixcountry there are 12 months, but each month consists of 6 weeks. The month are named the same way we do, from January to December, but in each month the weeks have different lengths. In the k-th month the weeks consist of 6^{k-1} days. What is the number of days of the spring (March, April, May together)?

2   Find the number of integers n between 1 and 2021 such that $2^n + 2^{n+3}$ is a perfect square.

3   The figure shows a line intersecting a square lattice. The area of some arising quadrilaterals are also indicated. What is the area of the region with the question mark?
`https://cdn.artofproblemsolving.com/attachments/0/d/4d5741a63d052e3f6971f87e60ca7df7302fb`
`png`

4   What is the number of 4-digit numbers that contains exactly 3 different digits that have consecutive value? Such numbers are for instance 5464 or 2001.

Two digits in base 10 are consecutive if their difference is 1.

5   How many integers $1 \le x \le 2021$ make the value of the expression

$$\frac{2x^3 - 6x^2 - 3x - 20}{5(x - 4)}$$

an integer?

6   Bertalan thought about a 4-digit positive number. Then he draw a simple graph on 4 vertices and wrote the digits of the number to the vertices of the graph in such a way that every vertex received exactly the degree of the vertex. In how many ways could he think about? In a simple graph every edge connects two different vertices, and between two vertices at most one edge can go.

7 Jimmy's garden has right angled triangle shape that lies on island of circular shape in such a way that the corners of the triangle are on the shore of the island. When he made fences along the garden he realized that the length of the shortest side is 36 meter shorter than the longest side, and third side required 48 meter long fence. In the middle of the garden he built a house of circular shape that has the largest possible size. Jimmy measured the distance between the center of his house and the center of the island. What is the square of this distance?

8 John found all real numbers p such that in the polynomial $g(x) = (x-1)^2(p+2x)^2$, the quadratic term has coefficient 2021. What is the sum of all of these values p?

9 On an 8×8 chessboard, a rook stands on the bottom left corner square. We want to move it to the upper right corner, subject to the following rules: we have to move the rook exactly 9 times, such that the length of each move is either 3 or 4. (It is allowed to mix the two lengths throughout the "journey".) How many ways are there to do this? In each move, the rook moves horizontally or vertically.

10 A triangle is given. Its side a is of length 20 cm, and its area is 125 cm^2. It is also known that one of the angles lying on side a is twice as large as the other one. We cut the triangle into two parts at the median belonging to side a. Then we move the so-obtained two parts towards each other, such that the two segments of side a remain on the same line (i.e., the line initially occupied by side a). We move the two parts towards each other until we first reach a moment when the common part of the two segments is of length 4 cm. What is the area of the so-obtained shape in cm^2? The so-obtained shape is the union of the two parts, which is a heptagon.
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11 Japanese businessman Rui lives in America and makes a living from trading cows. On Black Thursday he was selling his cows for 2000 dollars each (the cows were of the same price), but after the financial crash there were huge fluctuations in the market and Rui was forced to follow them with his pricing. Every day he doubled, halved, multiplied by five or divided by five the price from the previous day (even if it meant he had to give change in cents). At the same time he managed to follow the Japanese superstition, so that the integer part of the price in dollars never started with digit 4. On the day when Billy visited him to buy some cows the price of each cow was 80 dollars. What is the minimal number of days that could have passed since Black Thursday by then?

12 Billy let his herd freely. Enjoying their time the horses started to jump on the squares of a lattice of meadow that is infinite in both directions. Each horse can jump as follows: horizontally or vertically moves three, then turn to left and moves two. Naturally, under the jump a horse don't touch the ground. The horses are standing on squares that no two can meet by such a jump. How many horses does Billy have if their number is the maximum possible?
The figure below shows where a horse can jump to. Notice that there 4 places and not 8 like in

chess.
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13 The trapezoid $ABCD$ satisfies $AB \parallel CD$, $AB = 70$, $AD = 32$ and $BC = 49$. We also know that $\angle ABC = 3\angle ADC$. How long is the base CD?

14 How many functions $f : \{1, 2, ..., 16\} \to \{1, 2, ..., 16\}$ have the property that $f(f(x)) - 4x$ is divisible by 17 for all integers $1 \le x \le 16$?

15 King Albrecht founded a family. In the family everyone has exactly 8 children. The only, but really important rule is that among the grandchildren of any person at most x can be named Bela. (None of Albrecht's children is called Bela.) For which x is it possible that after a certain time each newborn in the family has at least one direct ancestor in the Royal family called Bela.
No two of Albrecht's descendants (including himself) have a common child.

16 Consider a table consisting of 2×7 squares. Each little square is surrounded by walls (each internal wall belongs to two squares). We would like to remove some internal walls to make it possible to get from any square to any other one without crossing walls. How many ways can we do this while removing the minimal possible number of internal walls?
The figure shows a possible configuration, the remaining walls are marked in red, the removed ones are marked in light pink. Two configurations are considered the same if the same walls are removed.
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– Category E+

– Day 1

1 same as Day 1 E3

2 same as Day 1 E4

3 On the evening of Halloween a group of n kids collected k bars of chocolate of the same type. At the end of the evening they wanted to divide the bars so that everybody gets the same amount of chocolate, and none of the bars is broken into more than two pieces. For which n and k is this possible?

4 same as Day 1 E5

5 Let n be a positive integer. Show that every divisors of $2n^2 - 1$ gives a different remainder after division by $2n$.

6	same as Day 1 E6
–	Day 2
1	Given a right angled triangle ABC in which $\angle ACB = 90^o$. Let D be an inner point of AB, and let E be an inner point of AC. It is known that $\angle ADE = 90^o$, and that the length of the segment AD is 8, the length of the segment DE is 15, and the length of segment CE is 3. What is the area of triangle ABC?
2	In a french village the number of inhabitants is a perfect square. If 100 more people moved in, then the number of people would be 1 bigger than a perfect square. If again 100 more people moved in, then the number of people would be a perfect square again. How many people lives in the village if their number is the least possible?
3	same as Day 2 E5
4	same as Day 2 E6
5	Joe, who is already feared by all bandits in the Wild West, would like to officially become a sheriff. To do that, he has to take a special exam where he has to demonstrate his talent in three different areas: tracking, shooting and lasso throwing. He successfully completes each task with a given probability, independently of each other. He passes the exam if he can complete at least two of the tasks successfully. Joe calculated that in case he starts with tracking and completes it successfully, his chance of passing the exam is 32%. If he starts with successful shooting, the chance of passing is 49%, whereas if he starts with successful lasso throwing, he passes with probability 52%. The overall probability of passing (calculated before the start of the exam) is $X/1000$. What is the value of X?
6	same as Day 2 E10
7	same as Day 2 E9
8	Benedek wrote the following 300 statements on a piece of paper. $2\vert1!$ $3\vert1!$ $3\vert2!$ $4\vert1!$ $4\vert2!$ $4\vert3!$ $5\vert1!$ $5\vert2!$ $5\vert3!$ $5\vert4!$ ... $24\vert1!$ $24\vert2!$ $24\vert3!$ $24\vert4!$ $24\vert23!$ $25\vert1!$ $25\vert2!$ $25\vert3!$ $25\vert4!$ $25\vert23!$ $25\vert24!$ How many true statements did Benedek write down? The symbol $-$ denotes divisibility, e.g. $6\vert4!$ means that 6 is a divisor of number $4!$.
9	same as Day 2 E11
10	Billy owns some really energetic horses. They are hopping around on points of the plane having two integer coordinates. Each horse can make the following types of jumps. They can hop to a

point obtained from their current position via a translation by vector $(15, 9)$, $(-9, 15)$, $(-15, -9)$ or $(9, -15)$.

The horses are now standing on lattice points such that no two can meet by making jumps as above. What is the maximal possible number of horses Billy can own?

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11   same as Day 2 E14

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12   same as Day 2 E13

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13   At a table tennis competition, every pair of players played each other exactly once. Every boy beat thrice as many boys as girls, and every girl was beaten by twice as many girls as boys. How many competitors were there, if we know that there were 10 more boys than girls?

There are no draws in table tennis, every match was won by one of the two players.

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14   same as Day 2 E15

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15   same as Day 2 E16

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16   The angles of a convex quadrilateral form an arithmetic sequence in clockwise order, and its side lengths also form an arithmetic sequence (but not necessarily in clockwise order). If the quadrilateral is not a square, and its shortest side has length 1, then its perimeter is $a + \sqrt{b}4$, where a and b are positive integers. What is the value of $a + b$?