Turkey EGMO TST 2016

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Day 1 February 11th

1 Prove that
$$x^4y + y^4z + z^4x + xyz(x^3 + y^3 + z^3) \geq (x + y + z)(3xyz - 1)$$
for all positive real numbers x, y, z.

2 In a simple graph, there are two disjoint set of vertices A and B where A has k and B has 2016 vertices. Four numbers are written to each vertex using the colors red, green, blue and black. There is no any edge at the beginning. For each vertex in A, we first choose a color and then draw all edges from this vertex to the vertices in B having a larger number with the chosen color. It is known that for each vertex in B, the set of vertices in A connected to this vertex are different. Find the minimal possible value of k.

3 Let X be a variable point on the side BC of a triangle ABC. Let B' and C' be points on the rays $[XB$ and $[XC$, respectively, satisfying $B'X = BC = C'X$. The line passing through X and parallel to AB' cuts the line AC at Y and the line passing through X and parallel to AC' cuts the line AB at Z. Prove that all lines YZ pass through a fixed point as X varies on the line segment BC.

Day 2 February 12th

4 In a convex pentagon, let the perpendicular line from a vertex to the opposite side be called an altitude. Prove that if four of the altitudes are concurrent at a point then the fifth altitude also passes through this point.

5 A sequence $a_1, a_2, \dots$ consisting of 1's and 0's satisfies for all $k > 2016$ that
$$a_k = 0 \quad \Longleftrightarrow \quad a_{k-1} + a_{k-2} + \cdots + a_{k-2016} > 23.$$
Prove that there exist positive integers N and T such that $a_k = a_{k+T}$ for all $k > N$.

6 Prove that for every square-free integer $n > 1$, there exists a prime number p and an integer m satisfying
$$p \mid n \quad \text{and} \quad n \mid p^2 + p \cdot m^p.$$