The problems from the CCA Math Bonanza held on 5/28/2016

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by trumpeter

– Individual Round

I1 Compute the integer
$$\frac{2^{\left(2^5-2\right)/5-1}-2}{5}.$$

2016 CCA Math Bonanza Individual Round#1

I2 Rectangle $ABCD$ has perimeter 178 and area 1848. What is the length of the diagonal of the rectangle?

2016 CCA Math Bonanza Individual Round#2

I3 Amanda has the list of even numbers $2, 4, 6, \dots 100$ and Billy has the list of odd numbers $1, 3, 5, \dots 99$. Carlos creates a list by adding the square of each number in Amanda's list to the square of the corresponding number in Billy's list. Daisy creates a list by taking twice the product of corresponding numbers in Amanda's list and Billy's list. What is the positive difference between the sum of the numbers in Carlos's list and the sum of the numbers in Daisy's list?

2016 CCA Math Bonanza Individual#3

I4 The three digit number $n = CCA$ (in base 10), where $C \neq A$, is divisible by 14. How many possible values for n are there?

2016 CCA Math Bonanza Individual#4

I5 Let ABC be a triangle with $AB = 3$, $BC = 4$, and $AC = 5$. If D is the projection from B onto AC, E is the projection from D onto BC, and F is the projection from E onto AC, compute the length of the segment DF.

2016 CCA Math Bonanza Individual#5

I6 Let a, b, c be non-zero real numbers. The lines $ax + by = c$ and $bx + cy = a$ are perpendicular and intersect at a point P such that P also lies on the line $y = 2x$. Compute the coordinates of point P.

2016 CCA Math Bonanza Individual#6

I7 Simon is playing chess. He wins with probability 1/4, loses with probability 1/4, and draws with probability 1/2. What is the probability that, after Simon has played 5 games, he has won strictly more games than he has lost?

2016 CCA Math Bonanza Individual#7

I8 Let $f(x) = x^2 + x + 1$. Determine the ordered pair (p, q) of primes satisfying $f(p) = f(q) + 242$.

2016 CCA Math Bonanza#8

I9 Let $P(X) = X^5 + 3X^4 - 4X^3 - X^2 - 3X + 4$. Determine the number of monic polynomials $Q(x)$ with integer coefficients such that $\frac{P(X)}{Q(X)}$ is a polynomial with integer coefficients. Note: a monic polynomial is one with leading coefficient 1 (so $x^3 - 4x + 5$ is one but not $5x^3 - 4x^2 + 1$ or $x^2 + 3x^3$).

2016 CCA Math Bonanza Individual#9

I10 Let ABC be a triangle with $AC = 28$, $BC = 33$, and $\angle ABC = 2\angle ACB$. Compute the length of side AB.

2016 CCA Math Bonanza#10

I11 How many ways are there to place 8 1s and 8 0s in a 4×4 array such that the sum in every row and column is 2?

1	0	0	1
0	1	1	0
0	1	1	0
1	0	0	1

2016 CCA Math Bonanza Individual#11

I12 Let f be a function from the set $X = \{1, 2, \dots, 10\}$ to itself. Call a partition (S, T) of X $f-balanced$ if for all $s \in S$ we have $f(s) \in T$ and for all $t \in T$ we have $f(t) \in S$. (A partition (S, T) is a pair of subsets S and T of X such that $S \cap T = \emptyset$ and $S \cup T = X$. Note that (S, T) and (T, S) are considered the same partition).

Let $g(f)$ be the number of $f-balanced$ partitions, and let m equal the maximum value of g over all functions f from X to itself. If there are k functions satisfying $g(f) = m$, determine $m + k$.

2016 CCA Math Bonanza Individual#12

I13 Let $P(x)$ be a polynomial with integer coefficients, leading coefficient 1, and $P(0) = 3$. If the polynomial $P(x)^2 + 1$ can be factored as a product of two non-constant polynomials with integer coefficients, and the degree of P is as small as possible, compute the largest possible value of $P(10)$.

2016 CCA Math Bonanza Individual#13

I14 Compute

$$\sum_{k=1}^{420} \gcd(k, 420).$$

2016 CCA Math Bonanza Individual#14

I15 Let ABC be a triangle with $AB = 5$, $AC = 12$ and incenter I. Let P be the intersection of AI and BC. Define ω_B and ω_C to be the circumcircles of ABP and ACP, respectively, with centers O_B and O_C. If the reflection of BC over AI intersects ω_B and ω_C at X and Y, respectively, then $\frac{O_BO_C}{XY} = \frac{PI}{IA}$. Compute BC.

2016 CCA Math Bonanza Individual#15

– Team Round

T1 It takes 3 rabbits 5 hours to dig 9 holes. It takes 5 beavers 36 minutes to build 2 dams. At this rate, how many more minutes does it take 1 rabbit to dig 1 hole than it takes 1 beaver to build 1 dam?

2016 CCA Math Bonanza Team#1

T2 Perry the painter wants to paint his floor, but he decides to leave a 1 foot border along the edges. After painting his floor, Perry notices that the area of the painted region is the same as the area of the unpainted region. Perry's floor measures a x b feet, where $a > b$ and both a and b are positive integers. Find all possible ordered pairs (a, b).

2016 CCA Math Bonanza Team#2

T3 Find the sum of all integers n not less than 3 such that the measure, in degrees, of an interior angle of a regular n-gon is an integer.

2016 CCA Math Bonanza Team#3

T4 In the *minesweeper* game below, each unopened square (for example, the one in the top left corner) is either empty or contains a mine. The other squares are empty and display the number of mines in the neighboring 8 squares (if this is 0, the square is unmarked). What is the minimum possible number of mines present on the field?

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`=\&rn=bWluZXN3ZWVwZXIucG5n`

2016 CCA Math Bonanza Team#4

T5 How many permutations of the word "ACADEMY" have that there exist two vowels that are separated by an odd distance? For example, the X and Y in XAY are separated by an even distance, while the X and Y in XABY are separated by an odd distance. Note: the vowels are A, E, I, O, and U. Y is **NOT** a vowel.

2016 CCA Math Bonanza Team#5

T6 Consider the polynomials $P\left(x\right)=16x^4+40x^3+41x^2+20x+16$ and $Q\left(x\right)=4x^2+5x+2$. If a is a real number, what is the smallest possible value of $\frac{P(a)}{Q(a)}$?

2016 CCA Math Bonanza Team#6

T7 A *cuboctahedron*, shown below, is a polyhedron with 8 equilateral triangle faces and 6 square faces. Each edge has the same length and each of the 24 vertices borders 2 squares and 2 triangles. An *octahedron* is a regular polyhedron with 6 vertices and 8 equilateral triangle faces. Compute the sum of the volumes of an octahedron with side length 5 and a cuboctahedron with side length 5.

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=\&rn=Q3Vib2N0YWhlZHJvbi5qcGc=

2016 CCA Math Bonanza Team#7

T8 As a, b and c range over *all* real numbers, let m be the smallest possible value of

$$2\left(a+b+c\right)^2+\left(ab-4\right)^2+\left(bc-4\right)^2+\left(ca-4\right)^2$$

and n be the number of ordered triplets (a,b,c) such that the above quantity is minimized. Compute $m+n$.

2016 CCA Math Bonanza Team#8

T9 Let ABC be a triangle with $AB=8$, $BC=9$, and $CA=10$. The line tangent to the circumcircle of ABC at A intersects the line BC at T, and the circle centered at T passing through A intersects the line AC for a second time at S. If the angle bisector of $\angle SBA$ intersects SA at P, compute the length of segment SP.

2016 CCA Math Bonanza Team#9

T10 Plusses and minuses are inserted in the expression

$$\pm 1 \pm 2 \pm 3 \dots \pm 2016$$

such that when evaluated the result is divisible by 2017. Let there be N ways for this to occur. Compute the remainder when N is divided by 503.

2016 CCA Math Bonanza Team#10

– Lightning Round

L1.1 What is the sum of all the integers n such that $|n-1|<\pi$?

2016 CCA Math Bonanza Lightning#1.1

L1.2 What is the largest prime factor of $729-64$?

2016 CCA Math Bonanza Lightning#1.2

L1.3 If the GCD of a and b is 12 and the LCM of a and b is 168, what is the value of $a\times b$?

2016 CCA Math Bonanza L1.3

L1.4 A triangle has a perimeter of 4 *yards* and an area of 6 square *feet*. If one of the angles of the triangle is right, what is the length of the largest side of the triangle, in feet?

2016 CCA Math Bonanza Lightning#1.4

L2.1 Bhairav runs a 15-mile race at 28 miles per hour, while Daniel runs at 15 miles per hour and Tristan runs at 10 miles per hour. What is the greatest length of time, in *minutes*, between consecutive runners' finishes?

2016 CCA Math Bonanza Lightning#2.1

L2.2 In triangle ABC, $AB=7$, $AC=9$, and $BC=8$. The angle bisector of $\angle BAC$ intersects side BC at D, and the angle bisector of $\angle ABC$ intersects AD at E. Compute AE^2.

2016 CCA Math Bonanza Lightning#2.2

L2.3 Let ABC be a right triangle with $\angle ACB=90^\circ$. D is a point on AB such that $CD\perp AB$. If the area of triangle ABC is 84, what is the smallest possible value of

$$AC^2+(3\cdot CD)^2+BC^2?$$

2016 CCA Math Bonanza Lightning#2.3

L2.4 What is the largest integer that must divide n^5-5n^3+4n for all integers n?

2016 CCA Math Bonanza Lightning#2.4

L3.1 How many 3-digit positive integers have the property that the sum of their digits is greater than the product of their digits?

2016 CCA Math Bonanza Lightning#3.1

L3.2 Let $a_0 = 1$ and define the sequence $\{a_n\}$ by

$$a_{n+1} = \frac{\sqrt{3}a_n - 1}{a_n + \sqrt{3}}.$$

If a_{2017} can be expressed in the form $a + b\sqrt{c}$ in simplest radical form, compute $a + b + c$.

2016 CCA Math Bonanza Lightning#3.2

L3.3 Triangle ABC has side length $AB = 5$, $BC = 12$, and $CA = 13$. Circle Γ is centered at point X exterior to triangle ABC and passes through points A and C. Let D be the second intersection of Γ with segment $\overline{BC}$. If $\angle BDA = \angle CAB$, the radius of Γ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

2016 CCA Math Bonanza Lightning#3.3

L3.4 Let S be the set of the reciprocals of the first 2016 positive integers and T the set of all subsets of S that form arithmetic progressions. What is the largest possible number of terms in a member of T?

2016 CCA Math Bonanza Lightning#3.4

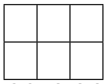
L4.1 Determine the remainder when

$$2^6 \cdot 3^{10} \cdot 5^{12} - 75^4\left(26^2 - 1\right)^2 + 3^{10} - 50^6 + 5^{12}$$

is divided by 1001.

2016 CCA Math Bonanza Lightning#4.1

L4.2 Consider the 2×3 rectangle below. We fill in the small squares with the numbers $1, 2, 3, 4, 5, 6$ (one per square). Define a *tasty* filling to be one such that each row is **not** in numerical order from left to right and each column is **not** in numerical order from top to bottom. If the probability that a randomly selected filling is tasty is $\frac{m}{n}$ for relatively prime positive integers m and n, what is $m + n$?



2016 CCA Math Bonanza Lightning#4.2

L4.3 Let ABC be a non-degenerate triangle with perimeter 4 such that $a = bc\sin^2 A$. If M is the maximum possible area of ABC and m is the minimum possible area of ABC, then $M^2 + m^2$ can be expressed in the form $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $a + b$.

2016 CCA Math Bonanza Lightning#4.3

L4.4 Real numbers $X_1, X_2, \ldots, X_{10}$ are chosen uniformly at random from the interval $[0, 1]$. If the expected value of $\min(X_1, X_2, \ldots, X_{10})^4$ can be expressed as a rational number $\frac{m}{n}$ for relatively prime positive integers m and n, what is $m + n$?

2016 CCA Math Bonanza Lightning#4.4

L5.1 The first question was asked in Set 4. The second question was asked in Set 5.

Question) Eshaan the Elephant has a long memory. He remembers that out of the integers $0, 1, 2, \ldots, 15$, one of them is special. Submit to the grader an ordered 4-tuple of subsets of $0, 1, 2, \ldots, 15$ and they will tell you whether the special number is in each. You can then submit your guess for the special number on the next round for points. (You might want to write down a copy of your submission somewhere other than your answer sheet. Note that this question itself is not worth any points, though the corresponding problem in Set 5 is.)

Question) Eshaan the Elephant has a long memory. He remembers that out of the integers $0, 1, 2, \ldots, 15$, one of them is special. You have submitted an ordered 4-tuple of subsets of $0, 1, 2, \ldots, 15$. Here is your reply from the grader.

1	2	3	4
Y/N	Y/N	Y/N	Y/N

What is the special number?

2016 CCA Math Bonanza Lightning#5.1

L5.2 In this problem, the symbol 0 represents the number zero, the symbol 1 represents the number seven, the symbol 2 represents the number five, the symbol 3 represents the number three, the symbol 4 represents the number four, the symbol 5 represents the number two, the symbol 6 represents the number nine, the symbol 7 represents the number one, the symbol 8 represents an arbitrarily large positive integer, the symbol 9 represents the number six, and the symbol ∞ represents the number eight. Compute the value of $\left|0 - 1 + 2 - 3^4 - 5 + 6 - 7^8 \times 9 - \infty\right|$.

2016 CCA Math Bonanza Lightning#5.2

L5.3 Let $A(x) = \lfloor \frac{x^2-20x+16}{4} \rfloor$, $B(x) = \sin\left(e^{\cos\sqrt{x^2+2x+2}}\right)$, $C(x) = x^3 - 6x^2 + 5x + 15$, $H(x) = x^4 + 2x^3 + 3x^2 + 4x + 5$, $M(x) = \frac{x}{2} - 2\lfloor \frac{x}{2} \rfloor + \frac{x}{2^2} + \frac{x}{2^3} + \frac{x}{2^4} + \ldots$, $N(x) = \text{the number of integers that divide } \lfloor x \rfloor$, $O(x) = |x| \log|x| \log\log|x|$, $T(x) = \sum_{n=1}^{\infty} \frac{n^x}{(n!)^3}$, and $Z(x) = \frac{x^{21}}{2016+20x^{16}+16x^{20}}$ for any real number x such that the functions are defined. Determine

$$C(C(A(M(A(T(H(B(O(N(A(N(Z(A(2016)))))))))))))).$$

2016 CCA Math Bonanza Lightning#5.3

L5.4 In the game of Colonel Blotto, you have 100 troops to distribute among 10 castles. Submit a 10-tuple $(x_1, x_2, \ldots x_{10})$ of nonnegative integers such that $x_1 + x_2 + \cdots + x_{10} = 100$, where each x_i represent the number of troops you want to send to castle i. Your troop distribution will be matched up against each opponent's and you will win 10 points for each castle that you send more troops to (if you send the same number, you get 5 points, and if you send fewer, you get none). Your aim is to score the most points possible averaged over all opponents.

For example, if team A submits $(90, 10, 0, \dots, 0)$, team B submits $(11, 11, 11, 11, 11, 11, 11, 11, 11, 1)$, and team C submits $(10, 10, 10, \dots 10)$, then team A will win 10 points against team B and 15 points against team C, while team B wins 90 points against team C. Team A averages 12.5 points, team B averages 90 points, and team C averages 47.5 points.

2016 CCA Math Bonanza Lightning#5.4