Stanford Mathematics Tournament 2018

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Team Round

– Short Answer Questions

– **p1.** Suppose A and B are points in the plane lying on the parabola $y = x^2$, and the x-coordinates of A and B are -29 and 51, respectively. Let C be the point where line AB intersects the y-axis. What is the y-coordinate of C?

p2. Cindy has a collection of identical rectangular prisms. She stacks them, end to end, to form 1 longer rectangular prism. Suppose that joining 11 of them will form a rectangular prism with 3 times the surface area of an individual rectangular prism. How many will she need to join end to end to form a rectangular prism with 9 times the surface area?

p3. A lattice point is a point (a, b) on the Cartesian plane where a and b are integers. Compute the number of lattice points in the interior and on the boundary of the triangle with vertices at $(0, 0)$, $(0, 20)$, and $(18, 0)$.

p4. Let $1 = a_1 < a_2 < a_3 < ... < a_k = n$ be the positive divisors of n in increasing order. If $n = a_3^3 - a_2^3$, what is n?

p5. A point (x_0, y_0) with integer coordinates is a primitive point of a circle if for some pair of integers (a, b), the line $ax + by = 1$ intersects the circle at (x_0, y_0). How many primitive points are there of the circle centered at $(2, -3)$ with radius 5?

p6. Three distinct points are chosen uniformly at random from the vertices of a regular 2018-gon. What is the probability that the triangle formed by these points is a right triangle?

p7. Consider any 5 points placed on the surface of a cube of side length 2 centered at the origin. Let m_x be the minimum distance between the x coordinates of any of the 5 points, m_y be the minimum distance between y coordinates, and m_z be the minimum distance between z coordinates. What is the maximum value of $m_x + m_y + m_z$?

p8. Eddy has two blank cubes A and B and a marker. Eddy is allowed to draw a total of 36 dots on cubes A and B to turn them into dice, where each side has an equal probability of appearing, and each side has at least one dot on it. Eddy then rolls dice A twice and dice B once and computes the product of the three numbers. Given that Eddy draws dots on the two dice to maximize his expected product, what is his expected product?

p9. Let $ABCD$ be a square. Point E is chosen inside the square such that $AE = 6$. Point F is chosen outside the square such that $BE = BF = 2\sqrt{5}$, $\angle ABF = \angle CBE$, and $AEBF$ is cyclic. Compute the area of $ABCD$.

p10. Find the total number of sets of nonnegative integers (w, x, y, z) where $w \le x \le y \le z$ such that $5w + 3x + y + z = 100$.

p11. Let $f(k)$ be a function defined by the following rules:
(a) $f(k)$ is multiplicative. In other words, if $gcd(a, b) = 1$, then $f(ab) = f(a) \cdot f(b)$,
(b) $f(p^k) = k$ for primes $p = 2, 3$ and all $k > 0$,
(c) $f(p^k) = 0$ for primes $p > 3$ and all $k > 0$, and
(d) $f(1) = 1$.
For example, $f(12) = 2$ and $f(160) = 0$. Evaluate $\sum_{k=1}^{\infty} \frac{f(k)}{k}$.

p12. Consider all increasing arithmetic progressions of the form $\frac{1}{a}$,$\frac{1}{b}$,$\frac{1}{c}$ such that $a, b, c \in N$ and $gcd(a, b, c) = 1$. Find the sum of all possible values of $\frac{1}{a}$.

p13. In $\triangle ABC$, let D, E, and F be the feet of the altitudes drawn from A, B, and C respectively. Let P and Q be points on line EF such that BP is perpendicular to EF and CQ is perpendicular to EF. If $PQ = 2018$ and $DE = DF + 4$, find DE.

p14. Let A and B be two points chosen independently and uniformly at random inside the unit circle centered at O. Compute the expected area of $\triangle ABO$.

p15. Suppose that a, b, c, d are positive integers satisfying

$$25ab + 25ac + b^2 = 14bc$$

$$4bc + 4bd + 9c^2 = 31cd$$

$$9cd + 9ca + 25d^2 = 95da$$

$$5da + 5db + 20a^2 = 16ab$$

Compute $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Proof Questions

1 Prove that if 7 divides a^2+b^2+1, then 7 does not divide $a+b$.

2 Consider a game played on the integers in the closed interval $[1,n]$. The game begins with some tokens placed in $[1,n]$. At each turn, tokens are added or removed from$[1,n]$ using the following rule: For each integer $k \in [1,n]$, if exactly one of $k-1$ and $k+1$ has a token, place a token at k for the next turn, otherwise leave k blank for the next turn.
We call a position *static* if no changes to the interval occur after one turn. For instance, the trivial position with no tokens is static because no tokens are added or removed after a turn (because there are no tokens). Find all non-trivial static positions.

3 Show that if A is a shape in the Cartesian coordinate plane with area greater than 1, then there are distinct points (a,b), (c,d) in A where $a-c=2x+5y$ and $b-d=x+3y$ where x,y are integers.

4 Let F_k denote the series of Fibonacci numbers shifted back by one index, so that $F_0=1$, $F_1=1$, and $F_{k+1}=F_k+F_{k-1}$. It is known that for any fixed $n \geq 1$ there exist real constants b_n, c_n such that the following recurrence holds for all $k \geq 1$:

$$F_{n\cdot(k+1)} = b_n \cdot F_{n\cdot k} + c_n \cdot F_{n\cdot(k-1)}.$$

Prove that $|c_n|=1$ for all $n \geq 1$.

5 Let $ABCD$ be a quadrilateral with sides AB, BC, CD, DA and diagonals AC, BD. Suppose that all sides of the quadrilateral have length greater than 1, and that the difference between any side and diagonal is less than 1. Prove that the following inequality holds

$$(AB+BC+CD+DA+AC+BD)^2 > 2|AC^3-BC^3|+2|BD^3-AD^3|-(AB+CD)^3$$

– Geometry Round

1 Consider a semi-circle with diameter AB. Let points C and D be on diameter AB such that CD forms the base of a square inscribed in the semicircle. Given that $CD=2$, compute the length of AB.

2 Let $ABCD$ be a trapezoid with AB parallel to CD and perpendicular to BC. Let M be a point on BC such that $\angle AMB = \angle DMC$. If $AB=3$, $BC=24$, and $CD=4$, what is the value of $AM+MD$?

3 Let ABC be a triangle and D be a point such that A and D are on opposite sides of BC. Give that $\angle ACD = 75^o$, $AC = 2$, $BD = \sqrt{6}$, and AD is an angle bisector of both $\triangle\ ABC$ and $\triangle\ BCD$, find the area of quadrilateral $ABDC$.

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4 Let $a_1, a_2, ..., a_{12}$ be the vertices of a regular dodecagon D_1 $(12$-gon). The four vertices a_1, a_4, a_7, a_{10} form a square, as do the four vertices a_2, a_5, a_8, a_{11} and a_3, a_6, a_9, a_{12}. Let D_2 be the polygon formed by the intersection of these three squares. If we let$[A]$ denotes the area of polygon A, compute $\frac{[D_2]}{[D_1]}$.

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5 In $\triangle\ ABC$, $\angle ABC = 75^o$ and $\angle BAC$ is obtuse. Points D and E are on AC and BC, respectively, such that $\frac{AB}{BC} = \frac{DE}{EC}$ and $\angle DEC = \angle EDC$. Compute $\angle DEC$ in degrees.

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6 In $\triangle\ AB$C, $AB = 3$, $AC = 6,$ and D is drawn on BC such that AD is the angle bisector of $\angle BAC$. D is reflected across AB to a point E, and suppose that AC and BE are parallel. Compute CE.

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7 Two equilateral triangles ABC and DEF, each with side length 1, are drawn in 2 parallel planes such that when one plane is projected onto the other, the vertices of the triangles form a regular hexagon $AFBDCE$. Line segments AE, AF, BF, BD, $CD,$ and CE are drawn, and suppose that each of these segments also has length 1. Compute the volume of the resulting solid that is formed.

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8 Let ABC be a right triangle with $\angle ACB = 90^o$, $BC = 16$, and $AC = 12$. Let the angle bisectors of $\angle BAC$ and $\angle ABC$ intersect BC and AC at D and E respectively. Let AD and BE intersect at I, and let the circle centered at I passing through C intersect AB at P and Q such that $AQ < AP$. Compute the area of quadrilateral $DPQE$.

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9 Let $ABCD$ be a cyclic quadrilateral with $3AB = 2AD$ and $BC = CD$. The diagonals AC and BD intersect at point X. Let E be a point on AD such that $DE = AB$ and Y be the point of intersection of lines AC and BE. If the area of triangle ABY is 5, then what is the area of quadrilateral $DEYX$?

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10 Let ABC be a triangle with $AB = 13$, $AC = 14$, and $BC = 15$, and let Γ be its incircle with incenter I. Let D and E be the points of tangency between Γ and BC and AC respectively, and let ω be the circle inscribed in $CDIE$. If Q is the intersection point between Γ and ω and P is the intersection point between CQ and ω, compute the length of PQ.

– Geometry Tiebreaker

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1 Point E is on side CD of rectangle $ABCD$ such that $\frac{CE}{DE} = \frac{2}{5}$. If the area of triangle BCE is 30, what is the area of rectangle $ABCD$?

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2 What is the largest possible height of a right cylinder with radius 3 that can fit in a cube with side length 12?

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3 A triangle has side lengths of 7, 8, and 9. Find the radius of the largest possible semicircle inscribed in the triangle.
