Regional Round of Durer Math Competition

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– Category E

1 Albrecht is travelling in his car on the motorway at a constant speed. The journey is very long so Marvin who is sitting next to Albrecht gets bored and decides to calculate the speed of the car. He was a bit careless but he noted that at noon they passed milestone XY (where X and Y are digits), at $12:42$ milestone YX and at 1pm they arrived at milestone $X0Y$. What did Marvin deduce, what is the speed of the car?

2 The best part of grandma's 18 cm $\times 36$ cm rectangle-shaped cake is the chocolate covering on the edges. Her three grandchildren would like to split the cake between each other so that everyone gets the same amount (of the area) of the cake, and they all get the same amount of the delicious perimeter too.
a) Can they cut the cake into three convex pieces like that?
b) The next time grandma baked this cake, the whole family wanted to try it so they had to cut the cake into six convex pieces this way. Is this possible?
c) Soon the entire neighbourhood has heard of the delicious cake. Can the cake be cut into 12 convex pieces with the same conditions?

3 The floor plan of a contemporary art museum is a (not necessarily convex) polygon and its walls are solid. The security guard guarding the museum has two favourite spots (points A and B) because one can see the whole area of the museum standing at either point. Is it true that from any point of the AB section one can see the whole museum?

4 Determine all triples of positive integers a, b, c that satisfy

a) $[a, b] + [a, c] + [b, c] = [a, b, c]$.

b) $[a, b] + [a, c] + [b, c] = [a, b, c] + (a, b, c)$.

Remark: Here $[x, y]$ denotes the least common multiple of positive integers x and y, and (x, y) denotes their greatest common divisor.

5 21 bandits live in the city of Warmridge, each of them having some enemies among the others. Initially each bandit has 240 bullets, and duels with all of his enemies. Every bandit distributes his bullets evenly between his enemies, this means that he takes the same number of bullets to each of his duels, and uses each of his bullets in only one duel. In case the number of his bullets is not divisible by the number of his enemies, he takes as many bullets to each duel as possible, but takes the same number of bullets to every duel, so it is possible that in the end the

bandit will have some remaining bullets.
Shooting is banned in the city, therefore a duel consists only of comparing the number of bullets in the guns of the opponents, and the winner is whoever has more bullets. After the duel the sheriff takes the bullets of the winner and as an act of protest the loser shoots all of his bullets into the air. What is the largest possible number of bullets the sheriff can have after all of the duels have ended?

Being someones enemy is mutual. If two opponents have the same number of bullets in their guns during a duel, then the sheriff takes the bullets of the bandit who has the wider hat among them.

Example: If a bandit has 13 enemies then he takes 18 bullets with himself to each duel, and they will have 6 leftover bullets after finishing all their duels.

– Category E+

1 same as E4

2 same as E5

3 Let k_1 and k_2 be two circles that are externally tangent at point C. We have a point A on k_1 and a point B on k_2 such that C is an interior point of segment AB. Let k_3 be a circle that passes through points A and B and intersects circles k_1 and k_2 another time at points M and N respectively. Let k_4 be the circumscribed circle of triangle CMN. Prove that the centres of circles k_1, k_2, k_3 and k_4 all lie on the same circle.

4 Find all pairs of polynomials (p, q) with integer coefficients that satisfy the equation

$$p(x^2) + q(x^2) = p(x)q(x)$$

such that p is of degree n and has n nonnegative real roots (with multiplicity).

5 There are n distinct lines in three-dimensional space such that no two lines are parallel and no three lines meet at one point. What is the maximal possible number of planes determined by these n lines?

We say that a plane is determined if it contains at least two of the lines.