Berkley mini Math Tournament Fall 2015

www.artofproblemsolving.com/community/c2780535
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Team Round **p1.** Let f be a function such that $f(x+y) = f(x) + f(y)$ for all x and y. Assume $f(5) = 9$. Compute $f(2015)$.

p2. There are six cards, with the numbers $2, 2, 4, 4, 6, 6$ on them. If you pick three cards at random, what is the probability that you can make a triangles whose side lengths are the chosen numbers?

p3. A train travels from Berkeley to San Francisco under a tunnel of length 10 kilometers, and then returns to Berkeley using a bridge of length 7 kilometers. If the train travels at 30 km/hr underwater and 60 km/hr above water, what is the train's average speed in km/hr on the round trip?

p4. Given a string consisting of the characters A, C, G, U, its reverse complement is the string obtained by first reversing the string and then replacing A's with U's, C's with G's, G's with C's, and U's with A's. For example, the reverse complement of UAGCAC is GUGCUA. A string is a palindrome if it's the same as its reverse. A string is called self-conjugate if it's the same as its reverse complement. For example, UAGGAU is a palindrome and UAGCUA is self-conjugate. How many six letter strings with just the characters A, C, G (no U's) are either palindromes or self-conjugate?

p5. A scooter has 2 wheels, a chair has 6 wheels, and a spaceship has 11 wheels. If there are 10 of these objects, with a total of 50 wheels, how many chairs are there?

p6. How many proper subsets of $\{1, 2, 3, 4, 5, 6\}$ are there such that the sum of the elements in the subset equal twice a number in the subset?

p7. A circle and square share the same center and area. The circle has radius 1 and intersects the square on one side at points A and B. What is the length of $\overline{AB}$?

p8. Inside a circle, chords AB and CD intersect at P in right angles. Given that $AP = 6$, $BP = 12$ and $CD = 15$, find the radius of the circle.

p9. Steven makes nonstandard checkerboards that have 29 squares on each side. The checkerboards have a black square in every corner and alternate red and black squares along every row and column. How many black squares are there on such a checkerboard?

p10. John is organizing a race around a circular track and wants to put 3 water stations at 9 possible spots around the track. He doesn't want any 2 water stations to be next to each other because that would be inefficient. How many ways are possible?

p11. In square $ABCD$, point E is chosen such that CDE is an equilateral triangle. Extend CE and DE to F and G on AB. Find the ratio of the area of $\triangle EFG$ to the area of $\triangle CDE$.

p12. Let S be the number of integers from 2 to 8462 (inclusive) which does not contain the digit $1, 3, 5, 7, 9$. What is S?

p13. Let x, y be non zero solutions to $x^2 + xy + y^2 = 0$. Find $\frac{x^{2016}+(xy)^{1008}+y^{2016}}{(x+y)^{2016}}$.

p14. A chess contest is held among 10 players in a single round (each of two players will have a match). The winner of each game earns 2 points while loser earns none, and each of the two players will get 1 point for a draw. After the contest, none of the 10 players gets the same score, and the player of the second place gets a score that equals to $4/5$ of the sum of the last 5 players. What is the score of the second-place player?

p15. Consider the sequence of positive integers generated by the following formula $a_1 = 3$, $a_{n+1} = a_n + a_n^2$ for $n = 2, 3, ...$
What is the tens digit of a_{1007}?

p16. Let (x, y, z) be integer solutions to the following system of equations $x^2z + y^2z + 4xy = 48$
$x^2 + y^2 + xyz = 24$
Find $\sum x + y + z$ where the sum runs over all possible (x, y, z).

p17. Given that $x + y = a$ and $xy = b$ and $1 \le a, b \le 50$, what is the sum of all a such that $x^4 + y^4 - 2x^2y^2$ is a prime squared?

p18. In $\triangle ABC$, M is the midpoint of $\overline{AB}$, point N is on side $\overline{BC}$. Line segments $\overline{AN}$ and $\overline{CM}$ intersect at O. If $AO = 12$, $CO = 6$, and $ON = 4$, what is the length of OM?

p19. Consider the following linear system of equations. $1+a+b+c+d = 1$ $16+8a+4b+2c+d = 2$

$81 + 27a + 9b + 3c + d = 3\ 256 + 64a + 16b + 4c + d = 4$
Find $a - b + c - d$.

p20. Consider flipping a fair coin 8 times. How many sequences of coin flips are there such that the string HHH never occurs?

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

Ind. Round **p1.** What is the units digit of $1 + 9 + 9^2 + ... + 9^{2015}$?

p2. In Fourtown, every person must have a car and therefore a license plate. Every license plate must be a 4-digit number where each digit is a value between 0 and 9 inclusive. However 0000 is not a valid license plate. What is the minimum population of Fourtown to guarantee that at least two people who have the same license plate?

p3. Two sides of an isosceles triangle $\triangle ABC$ have lengths 9 and 4. What is the area of $\triangle ABC$?

p4. Let x be a real number such that $10^{\frac{1}{x}} = x$. Find $(x^3)^{2x}$.

p5. A Berkeley student and a Stanford student are going to visit each others campus and go back to their own campuses immediately after they arrive by riding bikes. Each of them rides at a constant speed. They first meet at a place 17.5 miles away from Berkeley, and secondly 10 miles away from Stanford. How far is Berkeley away from Stanford in miles?

p6. Let $ABCDEF$ be a regular hexagon. Find the number of subsets S of $\{A, B, C, D, E, F\}$ such that every edge of the hexagon has at least one of its endpoints in S.

p7. A three digit number is a multiple of 35 and the sum of its digits is 15. Find this number.

p8. Thomas, Olga, Ken, and Edward are playing the card game SAND. Each draws a card from a 52 card deck. What is the probability that each player gets a di erent rank and a different suit from the others?

p9. An isosceles triangle has two vertices at $(1, 4)$ and $(3, 6)$. Find the x-coordinate of the third vertex assuming it lies on the x-axis.

p10. Find the number of functions from the set $\{1, 2, ..., 8\}$ to itself such that $f(f(x)) = x$ for all $1 \le x \le 8$.

p11. The circle has the property that, no matter how it's rotated, the distance between the highest and the lowest point is constant. However, surprisingly, the circle is not the only shape with that property. A Reuleaux Triangle, which also has this constant diameter property, is constructed as follows. First, start with an equilateral triangle. Then, between every pair of vertices of the triangle, draw a circular arc whose center is the 3rd vertex of the triangle. Find the ratio between the areas of a Reuleaux Triangle and of a circle whose diameters are equal.

p12. Let a, b, c be positive integers such that gcd $(a, b) = 2$, gcd $(b, c) = 3$, lcm $(a, c) = 42$, and lcm $(a, b) = 30$. Find abc.

p13. A point P is inside the square $ABCD$. If $PA = 5$, $PB = 1$, $PD = 7$, then what is PC?

p14. Find all positive integers n such that, for every positive integer x relatively prime to n, we have that n divides $x^2 - 1$. You may assume that if $n = 2^k m$, where m is odd, then n has this property if and only if both 2^k and m do.

p15. Given integers a, b, c satisfying

$$abc + a + c = 12$$

$$bc + ac = 8$$

$$b - ac = -2,$$

what is the value of a?

p16. Two sides of a triangle have lengths 20 and 30. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

p17. Find the number of non-negative integer solutions (x, y, z) of the equation

$$xyz + xy + yz + zx + x + y + z = 2014.$$

p18. Assume that A, B, C, D, E, F are equally spaced on a circle of radius 1, as in the figure below. Find the area of the kite bounded by the lines EA, AC, FC, BE.
https://cdn.artofproblemsolving.com/attachments/7/7/57e6e1c4ef17f84a7a66a65e2aa2ab9c7fd05
png

p19. A positive integer is called cyclic if it is not divisible by the square of any prime, and whenever $p < q$ are primes that divide it, q does not leave a remainder of 1 when divided by p. Compute the number of cyclic numbers less than or equal to 100.

p20. On an 8×8 chess board, a queen can move horizontally, vertically, and diagonally in any direction for as many squares as she wishes. Find the average (over all 64 possible positions of the queen) of the number of squares the queen can reach from a particular square (do not count the square she stands on).

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