2013 Spring Lexington Math Tournament

www.artofproblemsolving.com/community/c2792116

by parmenides51

– Individual Round

Individual p1. What is the smallest positive integer divisible by 20, 12, and 13?

p2. Two circles of radius 5 are placed in the plane such that their centers are 7 units apart. What is the largest possible distance between a point on one circle and a point on the other?

p3. In a magic square, all the numbers in the rows, columns, and diagonals sum to the same value. How many 2×2 magic squares containing the integers $\{1, 2, 3, 4\}$ are there?

p4. Ethan's sock drawer contains two pairs of white socks and one pair of red socks. Ethan picks two socks at random. What is the probability that he picks two white socks?

p5. The sum of the time on a digital clock is the sum of the digits displayed on the screen. For example, the sum of the time at $10 : 23$ would be 6. Assuming the clock is a 12 hour clock, what is the greatest possible positive difference between the sum of the time at some time and the sum of the time one minute later?

p6. Given the expression $1 \div 2 \div 3 \div 4$, what is the largest possible resulting value if one were to place parentheses $()$ somewhere in the expression?

p7. At a convention, there are many astronomers, astrophysicists, and cosmologists. At $first$, all the astronomers and astrophysicists arrive. At this point, $\frac{3}{5}$ of the people in the room are astronomers. Then, all the cosmologists come, so now, 30% of the people in the room are astrophysicists. What fraction of the scientists are cosmologists?

p8. At $10 : 00$ AM, a minuteman starts walking down a 1200-step stationary escalator at 40 steps per minute. Halfway down, the escalator starts moving up at a constant speed, while the minuteman continues to walk in the same direction and at the same pace that he was going before. At $10 : 55$ AM, the minuteman arrives back at the top. At what speed is the escalator going up, in steps per minute?

p9. Given that $x_1 = 57$, $x_2 = 68$, and $x_3 = 32$, let $x_n = x_{n-1} - x_{n-2} + x_{n-3}$ for $n \geq 4$. Find x_{2013}.

p10. Two squares are put side by side such that one vertex of the larger one coincides with a vertex of the smaller one. The smallest rectangle that contains both squares is drawn. If the area of the rectangle is 60 and the area of the smaller square is 24, what is the length of the diagonal of the rectangle?

p11. On a dield trip, 2 professors, 4 girls, and 4 boys are walking to the forest to gather data on butterflies. They must walk in a line with following restrictions: one adult must be the first person in the line and one adult must be the last person in the line, the boys must be in alphabetical order from front to back, and the girls must also be in alphabetical order from front to back. How many such possible lines are there, if each person has a distinct name?

p12. Flatland is the rectangle with vertices A, B, C, and D, which are located at $(0, 0)$, $(0, 5)$, $(5, 5)$, and $(5, 0)$, respectively. The citizens put an exact map of Flatland on the rectangular region with vertices $(1, 2)$, $(1, 3)$, $(2, 3)$, and $(2, 2)$ in such a way so that the location of A on the map lies on the point $(1, 2)$ of Flatland, the location of B on the map lies on the point $(1, 3)$ of Flatland, the location of C on the map lies on the point $(2, 3)$ of Flatland, and the location of D on the map lies on the point $(2, 2)$ of Flatland. Which point on the coordinate plane is thesame point on the map as where it actually is on Flatland?

p13. S is a collection of integers such that any integer x that is present in S is present exactly x times. Given that all the integers from 1 through 22 inclusive are present in S and no others are, what is the average value of the elements in S?

p14. In rectangle $PQRS$ with $PQ < QR$, the angle bisector of $\angle SPQ$ intersects $\overline{SQ}$ at point T and $\overline{QR}$ at U. If $PT : TU = 3 : 1$, what is the ratio of the area of triangle PTS to the area of rectangle $PQRS$?

p15. For a function $f(x) = Ax^2 + Bx + C$, $f(A) = f(B)$ and $A + 6 = B$. Find all possible values of B.

p16. Let α be the sum of the integers relatively prime to 98 and less than 98 and β be the sum of the integers not relatively prime to 98 and less than 98. What is the value of $\frac{\alpha}{\beta}$?

p17. What is the value of the series $\frac{1}{3} + \frac{3}{9} + \frac{6}{27} + \frac{10}{81} + \frac{15}{243} + ...$?

p18. A bug starts at $(0, 0)$ and moves along lattice points restricted to (i, j), where $0 \le i, j \le 2$. Given that the bug moves 1 unit each second, how many different paths can the bug take such that it ends at $(2, 2)$ after 8 seconds?

p19. Let $f(n)$ be the sum of the digits of n. How many different values of $n < 2013$ are there such that $f(f(f(n))) \neq f(f(n))$ and $f(f(f(n))) < 10$?

p20. Let A and B be points such that $\overline{AB} = 14$ and let ω_1 and ω_2 be circles centered at A and B with radii 13 and 15, respectively. Let C be a point on ω_1 and D be a point on ω_2 such that $\overline{CD}$ is a common external tangent to ω_1 and ω_2. Let P be the intersection point of the two circles that is closer to $\overline{CD}$. If M is the midpoint of $\overline{CD}$, what is the length of segment $\overline{PM}$?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Team Round

Team Round p1. Alan leaves home when the clock in his cardboard box says $7:35$ AM and his watch says $7:41$ AM. When he arrives at school, his watch says $7:47$ AM and the $7:45$ AM bell rings. Assuming the school clock, the watch, and the home clock all go at the same rate, how many minutes behind the school clock is the home clock?

p2. Compute
$$\left(\frac{2012^{2012-2013}+2013}{2013}\right) \times 2012.$$
Express your answer as a mixed number.

p3. What is the last digit of
$$2^{3^{4^{5^{6^{7^{8^{9^{\cdot^{\cdot^{\cdot^{2013}}}}}}}}}}}?$$

p4. Let $f(x)$ be a function such that $f(ab) = f(a)f(b)$ for all positive integers a and b. If $f(2) = 3$ and $f(3) = 4$, find $f(12)$.

p5. Circle X with radius 3 is internally tangent to circle O with radius 9. Two distinct points P_1 and P_2 are chosen on O such that rays $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are tangent to circle X. What is the length of line segment P_1P_2?

p6. Zerglings were recently discovered to use the same 24-hour cycle that we use. However, instead of making 12-hour analog clocks like humans, Zerglings make 24-hour analog clocks. On these special analog clocks, how many times during 1 Zergling day will the hour and minute hands be exactly opposite each other?

p7. Three Small Children would like to split up 9 different flavored Sweet Candies evenly, so that each one of the Small Children gets 3 Sweet Candies. However, three blind mice steal one of the Sweet Candies, so one of the Small Children can only get two pieces. How many fewer ways are there to split up the candies now than there were before, assuming every Sweet Candy is different?

p8. Ronny has a piece of paper in the shape of a right triangle ABC, where $\angle ABC = 90^o$, $\angle BAC = 30^o$, and $AC = 3$. Holding the paper fixed at A, Ronny folds the paper twice such that after the first fold, $\overline{BC}$ coincides with $\overline{AC}$, and after the second fold, C coincides with A. If Ronny initially marked P at the midpoint of $\overline{BC}$, and then marked P' as the end location of P after the two folds, find the length of $\overline{PP'}$ once Ronny unfolds the paper.

p9. How many positive integers have the same number of digits when expressed in base 3 as when expressed in base 4?

p10. On a 2×4 grid, a bug starts at the top left square and arbitrarily moves north, south, east, or west to an adjacent square that it has not already visited, with an equal probability of moving in any permitted direction. It continues to move in this way until there are no more places for it to go. Find the expected number of squares that it will travel on. Express your answer as a mixed number.

PS. You had better use hide for answers.

Hexagon Area Let ABC be a triangle and O be its circumcircle. Let A', B', C' be the midpoints of minor arcs AB, BC and CA respectively. Let I be the center of incircle of ABC. If $AB = 13$, $BC = 14$ and $AC = 15$, what is the area of the hexagon $AA'BB'CC'$?
Suppose $m\angle BAC = \alpha$, $m\angle CBA = \beta$, and $m\angle ACB = \gamma$.

p10. Let the incircle of ABC be tangent to AB, BC, and AC at J, K, L, respectively. Compute the angles of triangles JKL and $A'B'C'$ in terms of α, β, and γ, and conclude that these two triangles are similar.

p11. Show that triangle $AA'C'$ is congruent to triangle $IA'C'$. Show that $AA'BB'CC'$ has twice the area of $A'B'C'$.

p12. Let $r = JL/A'C'$ and the area of triangle JKL be S. Using the previous parts, determine the area of hexagon $AA'BB'CC'$ in terms of r and S.

p13. Given that the circumradius of triangle ABC is $65/8$ and that $S = 1344/65$, compute r and the exact value of the area of hexagon $AA'BB'CC'$.

PS. You had better use hide for answers.

– Theme Round

– 1 Apple Pi and Other Desserts

– *"In order to make an apple pie from scratch, you must first create the universe."* – Carl Sagan

p1. Surya decides to sell gourmet cookies at LMT. If he sells them for 25 cents each, he sells 36 cookies. For every 4 cents he raises the price of each cookie, he sells 3 fewer cookies. What is the smallest possible price, in cents, that Surya's jealous friends can change each cookie to so that Surya has no customers?

p2. Three French sorbets and four Italian gelatos cost 144 dollars. Six Italian gelatos and nine Florida sundaes cost 117 dollars. Seven French sorbets and 1 Florida sundae cost 229 dollars. If Arul wants to buy one of each type of ice cream, how much will it cost him?

p3. We call a number *delicious* if it has exactly 4 factors and is divisible by 2. How many numbers less than 99 are delicious?

p4. Charlie has 6 apple pies that form a ring. He starts with the first pie and puts on 10 scoops of whipped cream. Then, he moves to the next pie and, with a $1/2$ chance for each case, puts on either one less or one more scoop of whipped cream than the amount he put on the previous pie. He continues this process until he puts some whipped cream on the sixth pie. What is the probability that the number of scoops of whipped cream on the sixth pie differs from that on the first pie by exactly 1?

p5. Hao has 32 ounces of pure lemonade in his juice box. On the first day, he drinks an ounce and then refills the juice box with an ounce of Hater-ade and mixes the mixture thoroughly. On the second day, he drinks 2 oz of the juice and then refills the juice box with 2 oz of Hater-ade. On the third day, he drinks 3 oz and then refills the juice box with 3 oz of Hater-ade, and so on. Hao stops when the juice box has no more lemonade in it. How many ounces of Hater-ade did Hao drink?

PS. You should use hide for answers.

– 2 Game Theory

– ”*You have to learn the rules of the game. And then you have to play better than anyone else.*” – Albert Einstein

p6. Three standard six-sided dice are rolled and the values on the top faces are added. Let p be the probability of getting a total sum of 17 and q be the probability of getting a total sum of 18. Find p/q.

p7. Bill, Bob, and Ben write their favorite numbers on a sheet of paper. Bob points out that each number has its digits sum to 4, Bill points out that they are all divisible by the same prime number greater than 10, and Ben points out that none of them have 0's. Find the sum of Bill's, Bob's, and Ben's favorite numbers, given no two of them have the same favorite number.

p8. Sara has a 3×3 square tiled with alternating black and white colors (like a checkerboard). When she chooses a square, that square and the squares that share an edge with it switch colors. What is the minimum number of squares Sara needs to choose in order to cover the board with one color?

p9. A checker piece of radius 1 is placed at an arbitrary location on an infinitely large checkerboard of 1×1 squares . The checker piece covers n squares, either partially or completely. How many possible values for n are there?

p10. In a game of laser tag, a robot is in the center of a rectangular room with dimensions 6 meters by 8 meters. The robot has two perpendicular arms that can fire lasers straight outwards. The robot can fire the two lasers so long as they would hit the same wall if nothing blocked them. If a person is standing at a random location inside the room, what is the probability that she can be hit by a laser? The lasers do not bounce off walls.

PS. You should use hide for answers.

– 3 Dream Jobs

– ”*Choose a job you love, and you will never have to work a day in your life.*” – Confucius

p11. Darwin has a fruit stand at the market. Frank purchases 5 apples, and after a 5% tax, the price is $\$2.50$. Next, Rohil buys 3 apples. Assuming that everyone pays the 5% tax, how much did Rohil have to pay?

p12. In architecture class, Professor Radian wants to build a bridge over his circular pond with radius 4. He randomly chooses two points A and B on the circumference of the pond to be the endpoints of his new bridge. What is the probability that the length of the bridge is greater than $4\sqrt{3}$?

p13. Track stars Noah and Jonah run around a circular track. It takes Noah 3 minutes 20 seconds to run around the track, while it takes Jonah 3 minutes 45 seconds to run around the track. They start running around the track from the same spot going in the same direction. In how many minutes will the two of them be in the same spot along the track?

p14. Dan the Detective needs to decrypt a secret message intercepted from a clan of first graders. There are only 6 letters in the clan's alphabet, and Dan knows that the encryption code used by the clan is created by replacing every letter with another. For example, one encryption code may be $A \to B \to C \to D \to E \to F \to A$, where $A \to B$ means that every instance of A is replaced by a B. Note that a letter cannot be replaced by itself and no letter can replace multiple other letters. Given these conditions, how many different encryption codes can exist?

p15. The High Guardians of LHS have been assigned to protect a new, high priority room. A High Guardian's field of vision has a 360^o range so that it can see everything not directly blocked by a wall – in other words, one cannot see around walls – but once placed in the room, a High Guardian cannot move from that position. The room has 8 straight walls along its boundary and needs to be guarded so that every point in the entire room is in the field of vision of at least one High Guardian. What is the minimum number of High Guardians necessary for this task, no matter how the room is shaped?

PS. You should use hide for answers.

– Guts Round

– Round 1

p1. How many powers of 2 are greater than 3 but less than 2013?

p2. What number is equal to six greater than three times the answer to this question?

p3. Surya Cup-a-tea-raju goes to Starbucks Coffee to sip coffee out of a styrofoam cup. The cup is a cylinder, open on one end, with base radius 3 centimeters and height 10 centimeters. What is the exterior surface area of the styrofoam cup?

Round 2

p4. Andrew has two 6-foot-length sticks that he wishes to make into two of the sides of the entrance to his fort, with the ground being the third side. If he wants to make his entrance in the shape of a triangle, what is the largest area that he can make the entrance?

p5. Ethan and Devin met a fairy who told them "if you have less than 15 dollars, I will give you cake". If both had integral amounts of dollars, and Devin had 5 more dollars than Ethan, but only Ethan got cake, how many different amounts of money could Ethan have had?

p6. If $2012^x = 2013$, for what value of a, in terms of x, is it true that $2012^a = 2013^2$?

Round 3

p7. Find the ordered triple (L, M, T) of positive integers that makes the following equation true:

$$1+\cfrac{1}{L+\cfrac{1}{M+\cfrac{1}{T}}}=\frac{79}{43}.$$

p8. Jonathan would like to start a banana plantation so he is saving up to buy an acre of land, which costs $\$600,000$. He deposits $\$300,000$ in the bank, which gives 20% interest compounded at the end of each year. At this rate, how many years will Jonathan have to wait until he can buy the acre of land?

p9. Arul and Ethan went swimming at their town pool and started to swim laps to see who was in better shape. After one hour of swimming at their own paces, Ethan completed 32 more laps than Arul. However, after that, Ethan got tired and swam at half his original speed while Arul's speed didn't change. After one more hour, Arul swam a total of 320 laps. How many laps did Ethan swim after two hours?

Round 4

p10. A right triangle with a side length of 6 and a hypotenuse of 10 has circles of radius 1 centered at each vertex. What is the area of the space inside the triangle but outside all three circles?

p11. In isosceles trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$ and the lengths of $\overline{AB}$ and $\overline{CD}$ are 2 and 6, respectively. Let the diagonals of the trapezoid intersect at point E. If the distance from E to $\overline{CD}$ is 9, what is the area of triangle ABE?

p12. If 144 unit cubes are glued together to form a rectangular prism and the perimeter of the base is 54 units, what is the height?

PS. You should use hide for answers. Rounds 6-8 are here (`https://artofproblemsolving.com/community/c3h3136014p28427163`) and 9-12 here (`https://artofproblemsolving.com/community/c3h3137069p28442224`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 5

p13. Given that $x^3 + y^3 = 208$ and $x + y = 4$, what is the value of $\frac{1}{x}+\frac{1}{y}$?

p14. Find the sum of all three-digit integers n such that the value of n is equal to the sum of the factorials of n's digits.

p15. Three christmas lights are initially off. The Grinch decides to fiddle around with the lights, switching one of the lights each second. He wishes to get every possible combination of lights. After how many seconds can the Grinch complete his task?

Round 6

p16. A regular tetrahedron of side length 1 has four similar tetrahedrons of side length $1/2$ chopped off, one from each of the four vertices. What is the sum of the numbers of vertices, edges, and faces of the remaining solid?

p17. Mario serves a pie in the shape of a regular 2013-gon. To make each slice, he must cut in a straight line starting from one vertex and ending at another vertex of the pie. Every vertex of a slice must be a vertex of the original 2013-gon. If every person eats at least one slice of pie regardless of the size, what is the maximum number of people the 2013-gon pie can feed?

p18. Find the largest integer x such that $x^2 + 1$ divides $x^3 + x - 1000$.

Round 7

p19. In $\triangle ABC$, $\angle B = 87^o$, $\angle C = 29^o$, and $AC = 37$. The perpendicular bisector of $\overline{BC}$ meets $\overline{AC}$ at point T. What is the value of $AB + BT$?

p20. Consider the sequence $f(1) = 1$, $f(2) = \frac{1}{2}$,$f(3) = \frac{1+3}{2}$, $f(4) = \frac{1+3}{2+4}$,$f(5) = \frac{1+3+5}{2+4}$... What is the minimum value of n, with $n > 1$, such that $|f(n) - 1| \le \frac{1}{10}$.

p21. Three unit circles are centered at $(0, 0)$,$(0, 2)$, and $(2, 0)$. A line is drawn passing through $(0, 1)$ such that the region inside the circles and above the line has the same area as the region inside the circles and below the line. What is the equation of this line in $y = mx + b$ form?

Round 8

p22. The two walls of a pinball machine are positioned at a 45 degree angle to each other. A pinball, represented by a point, is fired at a wall (but not at the intersection of the two walls). What is the maximum number of times the ball can bounce off the walls?

p23. Albert is fooling people with his weighted coin at a carnival. He asks his guests to guess how many times heads will show up if he flips the coin 4 times. Richard decides to play the game and guesses that heads will show up 2 times. In the previous game, Zach guessed that the heads would show up 3 times. In Zach's game, there were least 3 heads, and given this information, Zach had a $\frac{4}{9}$ chance of winning. What is the probability that Richard guessescorrectly?

p24. Let S be the set of all positive integers relatively prime to 2013 that have no prime factor greater than 15. Find the sum of the reciprocals of all of the elements of S.

PS. You should use hide for answers.Rounds 1-4 are here (`https://artofproblemsolving.com/community/c3h3134546p28406927`) and 9-12 here (`https://artofproblemsolving.com/community/c3h3137069p28442224`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 9

p25. Define a hilly number to be a number with distinct digits such that when its digits are read from left to right, they strictly increase, then strictly decrease. For example, 483 and 1230 are both hilly numbers, but 123 and 1212 are not. How many 5-digit hilly numbers are there?

p26. Triangle ABC has $AB = 4$ and $AC = 6$. Let the intersection of the angle bisector of $\angle BAC$

and $\overline{BC}$ be D and the foot of the perpendicular from C to the angle bisector of $\angle BAC$ be E. What is the value of AD/AE?

p27. Given that $(7+4\sqrt{3})^x+(7-4\sqrt{3})^x=10$, find all possible values of $(7+4\sqrt{3})^x-(7-4\sqrt{3})^x$.

Round 10

Note: In this set, the answers for each problem rely on answers to the other problems.

p28. Let X be the answer to question 29. If $5A+5B=5X-8$ and $A^2+AB-2B^2=0$, find the sum of all possible values of A.

p29. Let W be the answer to question 28. In isosceles trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, line segments $\overline{AC}$ and $\overline{BD}$ split each other in the ratio $2:1$. Given that the length of BC is W, what is the greatest possible length of $\overline{AB}$ for which there is only one trapezoid $ABCD$ satisfying the given conditions?

p30. Let W be the answer to question 28 and X be the answer to question 29. For what value of Z is $|Z-X|+|Z-W|-|W+X-Z|$ at a minimum?

Round 11

p31. Peijin wants to draw the horizon of Yellowstone Park, but he forgot what it looked like. He remembers that the horizon was a string of 10 segments, each one either increasing with slope 1, remaining flat, or decreasing with slope 1. Given that the horizon never dipped more than 1 unit below or rose more than 1 unit above the starting point and that it returned to the starting elevation, how many possible pictures can Peijin draw?

p32. DNA sequences are long strings of A, T, C, and G, called base pairs. (e.g. AATGCA is a DNA sequence of 6 base pairs). A DNA sequence is called stunningly nondescript if it contains each of A, T, C, G, in some order, in 4 consecutive base pairs somewhere in the sequence. Find the number of stunningly nondescript DNA sequences of 6 base pairs (the example above is to be included in this count).

p33. Given variables s, t that satisfy $(3+2s+3t)^2+(7-2t)^2+(5-2s-t)^2=83$, find the minimum possible value of $(-5+2s+3t)^2+(3-2t)^2+(2-2s-t)^2$.

Round 12

p34. Let $f(n)$ be the number of powers of 2 with n digits. For how many values of n from 1 to 2013 inclusive does $f(n) = 3$? If your answer is N and the actual answer is C, then the score you will receive on this problem is $max\{15 - \frac{|N-C|}{26039}, 0\}$, rounded to the nearest integer.

p35. How many total characters are there in the source files for the LMT 2013 problems? If your answer is N and the actual answer is C, then the score you receive on this problem is $max\{15 - \frac{|N-C|}{1337}, 0\}$, rounded to the nearest integer.

p36. Write down two distinct integers between 0 and 300, inclusive. Let S be the collection of everyone's guesses. Let x be the smallest nonnegative difference between one of your guesses and another guess in S (possibly your other guess). Your team will be awarded $min(15, x)$ points.

PS. You should use hide for answers.Rounds 1-4 are here (https://artofproblemsolving.com/community/c3h3134546p28406927) and 6-8 here (https://artofproblemsolving.com/community/c3h3136014p28427163). Collected here (https://artofproblemsolving.com/community/c5h2760506p24