2015 Spring Lexington Math Tournament
www.artofproblemsolving.com/community/c2801166
by parmenides51

– Individual Round

Individual p1. What is $\sqrt[2015]{2^01^5}$?

p2. What is the ratio of the area of square $ABCD$ to the area of square $ACEF$?

p3. 2015 in binary is 11111011111, which is a palindrome. What is the last year which also had this property?

p4. What is the next number in the following geometric series: 1020100, 10303010, 104060401?

p5. A circle has radius A and area r. If $A = r^2\pi$, then what is the diameter, C, of the circle?

p6. If

$$O + N + E = 1$$
$$T + H + R + E + E = 3$$
$$N + I + N + E = 9$$
$$T + E + N = 10$$
$$T + H + I + R + T + E + E + N = 13$$

Then what is the value of O?

p7. By shifting the initial digit, which is 6, of the positive integer N to the end (for example, 65 becomes 56), we obtain a number equal to $\frac{N}{4}$. What is the smallest such N?

p8. What is $\sqrt[3]{\frac{2015!(2013!)+2014!(2012!)}{2013!(2012!)}}$?

p9. How many permutations of the digits of 1234 are divisible by 11?

p10. If you choose 4 cards from a normal 52 card deck (with replacement), what is the probability that you will get exactly one of each suit (there are 4 suits)?

p11. If LMT is an equilateral triangle, and $MATH$ is a square, such that point A is in the triangle, then what is HL/AL?

p12. If

$$\begin{array}{cccccccc} & & & & & L & H & S \\ + & & & & H & I & G & H \\ + & & S & C & H & O & O & L \\ \hline = & & S & O & C & O & O & L \end{array}$$

and $\{M, A, T, H, S, L, O, G, I, C\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then what is the ordered pair $(M + A + T + H, [T + e + A + M])$ where e is 2.718...and $[n]$ is the greatest integer less than or equal to n ?

p13. There are 5 marbles in a bag. One is red, one is blue, one is green, one is yellow, and the last is white. There are 4 people who take turns reaching into the bag and drawing out a marble without replacement. If the marble they draw out is green, they get to draw another marble out of the bag. What is the probability that the 3rd person to draw a marble gets the white marble?

p14. Let a ”palindromic product” be a product of numbers which is written the same when written back to front, including the multiplication signs. For example, $234 * 545 * 432$, $2 * 2 * 2 * 2$, and $14 * 41$ are palindromic products whereas $2 * 14 * 4 * 12$, $567 * 567$, and $2 * 2 * 3 * 3 * 2$ are not. 2015 can be written as a ”palindromic product” in two ways, namely $13 * 5 * 31$ and $31 * 5 * 13$. How many ways can you write 2016 as a palindromic product without using 1 as a factor?

p15. Let a sequence be defined as $S_n = S_{n-1} + 2S_{n-2}$, and $S_1 = 3$ and $S_2 = 4$. What is $\sum_{n=1}^{\infty} \frac{S_n}{3^n}$?

p16. Put the numbers $0-9$ in some order so that every 2-digit substring creates a number which is either a multiple of 7, or a power of 2.

p17. Evaluate $\dfrac{8+\dfrac{8+\frac{8+...}{3+...}}{3+\frac{8+...}{3+...}}}{3+\dfrac{8+\frac{8+...}{3+...}}{3+\frac{8+...}{3+...}}}$, assuming that it is a positive real number.

p18. 4 non-overlapping triangles, each of area A, are placed in a unit circle. What is the maximum value of A?

p19. What is the sum of the reciprocals of all the (positive integer) factors of 120 (including 1 and 120 itself).

p20. How many ways can you choose 3 distinct elements of $\{1, 2, 3, ..., 4000\}$ to make an increasing arithmetic series?

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Team Round

Team Round The answers to each of the ten questions in this section are integers containing only the digits 1 through 8, inclusive. Each answer can be written into the grid on the answer sheet, starting from the cell with the problem number, and continuing across or down until the entire answer has been written. Answers may cross dark lines. If the answers are correctly filled in, it will be uniquely possible to write an integer from 1 to 8 in every cell of the grid, so that each number will appear exactly once in every row, every column, and every marked 2 by 4 box. You will get 7 points for every correctly filled answer, and a 15 point bonus for filling in every gridcell. It will help to work back and forth between the grid and the problems, although every problem is uniquely solvable on its own.
Please write clearly within the boxes. No points will be given for a cell without a number, with multiple
numbers, or with illegible handwriting.
https://cdn.artofproblemsolving.com/attachments/9/b/f4db097a9e3c2602b8608be64f06498bd9d58
png

1 ACROSS: Jack puts 10 red marbles, 8 green marbles and 4 blue marbles in a bag. If he takes out 11 marbles, what is the expected number of green marbles taken out?

2 DOWN: What is the closest integer to $6\sqrt{35}$?

3 ACROSS: Alan writes the numbers 1 to 64 in binary on a piece of paper without leading zeroes. How many more times will he have written the digit 1 than the digit 0?

4 ACROSS: Integers a and b are chosen such that $-50 < a, b \le 50$. How many ordered pairs (a, b) satisfy the below equation?

$$(a + b + 2)(a + 2b + 1) = b$$

5 DOWN: Zach writes the numbers 1 through 64 in binary on a piece of paper without leading zeroes. How many times will he have written the two-digit sequence ”10”?

6 ACROSS: If you are in a car that travels at 60 miles per hour, $\$1$ is worth 121 yen, there are 8 pints in a gallon, your car gets 10 miles per gallon, a cup of coffee is worth $\$2$, there are 2 cups in a pint, a gallon of gas costs $\$1.50$, 1 mile is about 1.6 kilometers, and you are going to a coffee shop 32 kilometers away for a gallon of coffee, how much, in yen, will it cost?

7 DOWN: Clive randomly orders the letters of ”MIXING THE LETTERS, MAN”. If $\frac{p}{m^n q}$ is the probability that he gets ”LMT IS AN EXTREME THING” where p and q are odd integers, and m is a prime number, then what is $m+n$?

8 ACROSS: Joe is playing darts. A dartboard has scores of $10, 7$, and 4 on it. If Joe can throw 12 darts, how many possible scores can he end up with?

9 ACROSS: What is the maximum number of bounded regions that 6 overlapping ellipses can cut the plane into?

10 DOWN: Let ABC be an equilateral triangle, such that A and B both lie on a unit circle with center O. What is the maximum distance between O and C? Write your answer be in the form $\frac{a\sqrt{b}}{c}$ where b is not divisible by the square of any prime, and a and c share no common factor. What is abc ?

PS. You had better use hide for answers.

– Theme Round

1 Songs

– **p1.** Meghan Trainor is all about those base-systems. If she weighs 451 pounds in some base, and 127 pounds in a base that is twice as big, how much does she weigh in base 10?

p2. Taylor Swift made the song 15 in 2008, and the song 22 in 2012. If she had continued at the same rate, in what year would her song 1989 have come out?

p3. Sir Mix-A-Lot likes big butts and cannot lie. He is with his three friends, who all always lie. If each of the following statements was said by a different person, who is Sir Mix-A-Lot?
A: B and D both like big butts.
B: C and D either both like big butts, or neither of them do.

C: B does not like big butts.
D: A or C (or both) like big butts.

p4. Mark Ronson is going to give some uptown funk-tions to you. Bruno Mars is at 1 degree Fahrenheit right now (that is, time, in seconds, $t = 0$), and his heat h(t) is given by $h(t) = 2 \cdot h(t-1) + t$. If he will explode when he is over $1,000,000$ degrees Fahrenheit, at what time t will Bruno Mars explode?

p5. Jay-Z, who is a emcee, took 4 AMC contests, for a total of 100 problems. He got 99 problems correct and 1 problem wrong. His answer, z, to the problem he missed, was the square root of the correct 4-digit answer, j. If z is also 3 times the sum of the digits of j, then what is the ordered tuple (j, z)?

PS. You had better use hide for answers.

2 Physics

– **p1.** Two buildings are connected by a rope. The rope is 20 meters long and each end is connected to the top corner of a building. If the rope droops 10 meters below the roofs of the buildings, then how far away are the buildings from each other?

p2. A ball bounces up and down and bounces up 80% of its height after each bounce. If it is dropped from a height of 2 feet above the ground, what is the total distance the ball will travel before coming to rest on the ground?

p3. The formula for the gravitational acceleration on a planet is $\frac{Gm}{r^2}$ where G is a constant, m is the mass of the planet, and r is the radius of the planet. If a certain planet has half the radius of Earth, and a third of the mass, what is the ratio of Earth's gravitational constant to that planet's?

p4. Albert knows that there are 6 types of leptons and 6 types of quarks, but he doesn't know the name of any of them. On a physics test, Albert is given a list of the names of all 12 particles, and has to label 6 as leptons and 6 as quarks. What is the probability that he gets at least 10 of the 12 particles correctly on the test?

p5. 3 planets are orbiting around a sun in coplanar orbits. One planet makes 1 orbit every 21 years, one makes an orbit every 33 years, and one makes an orbit every 77 years. If all of the planets and the sun lie along the same line right now, how long, in years, will it be before all 4 lie along the same line again?

PS. You had better use hide for answers.

3 Frisbees

– In the following problems, some number of kids are evenly spaced around the perimeter of a unit circle. One frisbee is being thrown from kid to kid, taking the path of the line in between them. To throw the frisbee "5 kids left", for example, would mean to throw it in a straight line from the current kid to one 5 spaces in the clockwise direction, skipping over 4 kids.

p1. There are 2015 kids, and each time the frisbee is thrown, it is thrown 100 kids left. How many kids will be guaranteed never be able to touch the frisbee?

p2. Call the probability that a pass made by a certain player is not caught, that player's "Likelihood of a Missed Throw", or LMT for short. If Ivan, who is standing in a circle with 9 other players, has an LMT of 90%, and each player's LMT is the average of the LMTs of the people on their left and right, then what is the likelihood that the frisbee will make it all the way back around to Ivan?

p3. If Zach threw the frisbee to his friend Clive, who is standing 3 spaces to Zach's left, the frisbee would go half as far as if he threw it to Henry, who is standing 9 spaces to Zach's left. How far would the frisbee go if Clive were to throw it to Henry?

p4. Throwing the frisbee a distance d takes a total time of d^2 seconds. Moreover, after a person catches a frisbee, it takes them one second before they can throw it. Suppose Steven is in a circle with 12 total people, and starts by throwing the frisbee. It goes clockwise around the circle back to Steven in N throws, with him catching it after 11 seconds. What is the sum of all the possible values of N?

p5. If there are 187 kids in the circle, and each kid will either pass the frisbee 17 spaces to their left, or 11 spaces to their right, then how many ways are there to pass the frisbee 187 times, such that every person has thrown the frisbee once and every person has caught the frisbee once?

PS. You had better use hide for answers.

– Guts Round

– Round 1

p1. Every angle of a regular polygon has degree measure 179.99 degrees. How many sides does it have?

p2. What is $\frac{1}{20}+\frac{1}{1}+\frac{1}{5}$?

p3. If the area bounded by the lines $y=0$, $x=0$, and $x=3$ and the curve $y=f(x)$ is 10 units, what is the area bounded by $y=0$, $x=0$, $x=6$, and $y=f(x/2)$?

Round 2

p4. How many ways can 42 be expressed as the sum of 2 or more consecutive positive integers?

p5. How many integers less than or equal to 2015 can be expressed as the sum of 2 (not necessarily distinct) powers of two?

p6. p, q, and q^2-p^2 are all prime. What is pq?

Round 3

p7. Let $p(x)=x^2+ax+a$ be a polynomial with integer roots, where a is an integer. What are all the possible values of a?

p8. In a given right triangle, the perimeter is 30 and the sum of the squares of the sides is 338. Find the lengths of the three sides.

p9. Each of the 6 main diagonals of a regular hexagon is drawn, resulting in 6 triangles. Each of those triangles is then split into 4 equilateral triangles by connecting the midpoints of the 3 sides. How many triangles are in the resulting figure?

Round 4

p10. Let $f=5x+3y$, where x and y are positive real numbers such that xy is 100. Find the minimum possible value of f.

p11. An integer is called ”Awesome” if its base 8 expression contains the digit string 17 at any point (i.e. if it ever has a 1 followed immediately by a 7). How many integers from 1 to 500 (base 10) inclusive are Awesome?

p12. A certain pool table is a rectangle measuring 15×24 feet, with 4 holes, one at each vertex. When playing pool, Joe decides that a ball has to hit at least 2 sides before getting into a hole or else the shot does not count. What is the minimum distance a ball can travel after being hit on this table if it was hit at a vertex (assume it only stops after going into a hole) such that the shot counts?

PS. You should use hide for answers. Rounds 5-8 have been posted here (`https://artofproblemsolving.com/community/c3h3157013p28696685`) and 9-12 here (`https://artofproblemsolving.com/community/c3h3158564p28715928`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 5

p13. Sally is at the special glasses shop, where there are many different optical lenses that distort what she sees and cause her to see things strangely. Whenever she looks at a shape through lens A, she sees a shape with 2 more sides than the original (so a square would look like a hexagon). When she looks through lens B, she sees the shape with 3 fewer sides (so a hexagon would look like a triangle). How many sides are in the shape that has 200 more diagonals when looked at from lense A than from lense B?

p14. How many ways can you choose 2 cells of a 5 by 5 grid such that they aren't in the same row or column?

p15. If $a + \frac{1}{b} = (2015)^{-1}$ and $b + \frac{1}{a} = (2016)^2$ then what are all the possible values of b?

Round 6

p16. In Canadian football, linebackers must wear jersey numbers from $30 - 35$ while defensive linemen must wear numbers from $33 - 38$ (both intervals are inclusive). If a team has 5 linebackers and 4 defensive linemen, how many ways can it assign jersey numbers to the 9 players such that no two people have the same jersey number?

p17. What is the maximum possible area of a right triangle with hypotenuse 8?

p18. 9 people are to play touch football. One will be designated the quarterback, while the other eight will be divided into two (indistinct) teams of 4. How many ways are there for this to be done?

Round 7

p19. Express the decimal 0.3 in base 7.

p20. 2015 people throw their hats in a pile. One at a time, they each take one hat out of the pile so that each has a random hat. What is the expected number of people who get their own hat?

p21. What is the area of the largest possible trapezoid that can be inscribed in a semicircle of radius 4?

Round 8

p22. What is the base 7 expression of $1211_3 \cdot 1110_2 \cdot 292_{11} \cdot 20_3$?

p23. Let $f(x)$ equal the ratio of the surface area of a sphere of radius x to the volume of that same sphere. Let $g(x)$ be a quadratic polynomial in the form $x^2 + bx + c$ with $g(6) = 0$ and the minimum value of $g(x)$ equal to c. Express $g(x)$ as a function of $f(x)$ (e.g. in terms of $f(x)$).

p24. In the country of Tahksess, the income tax code is very complicated. Citizens are taxed 40% on their first $\$20,000$ and 45% on their next $\$40,000$ and 50% on their next $\$60,000$ and so on, with each 5% increase in tax rate a ecting $\$20,000$ more than the previous tax rate. The maximum tax rate, however, is 90%. What is the overall tax rate (percentage of money owed) on 1 million dollars in income?

PS. You should use hide for answers. Rounds 1-4 have been posted here (https://artofproblemsolving.com/community/c3h3157009p28696627) and 9-12 here (https://artofproblemsolving.com/community/c3h3158564p28715928). .Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Round 9

p25. For how many nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ is the sum of the elements divisible by 32?

p26. America declared independence in 1776. Take the sum of the cubes of the digits of 1776

and let that equal S_1. Sum the cubes of the digits of S_1 to get S_2. Repeat this process 1776 times. What is S_{1776}?

p27. Every Golden Grahams box contains a randomly colored toy car, which is one of four colors. What is the expected number of boxes you have to buy in order to obtain one car of each color?

Round 10

p28. Let B be the answer to Question 29 and C be the answer to Question 30. What is the sum of the square roots of B and C?

p29. Let A be the answer to Question 28 and C be the answer to Question 30. What is the sum of the sums of the digits of A and C?

p30. Let A be the answer to Question 28 and B be the answer to Question 29. What is $A + B$?

Round 11

p31. If $x + \frac{1}{x} = 4$, find $x^6 + \frac{1}{x^6}$.

p32. Given a positive integer n and a prime p, there is are unique nonnegative integers a and b such
that $n = p^b \cdot a$ and $gcd(a, p) = 1$. Let $v_p(n)$ denote this uniquely determined a. Let S denote the set of the first 20 primes. Find $\sum_{p\in S} v_p \left(1 + \sum_{i=0}^{100} p^i\right)$.

p33. Find the maximum value of n such that $n + \sqrt{(n-1) + \sqrt{(n-2) + ... + \sqrt{1}}} < 49$ (Note: there would be $n - 1$ square roots and n total terms).

Round 12

p34. Give two numbers a and b such that $2015^a < 2015! < 2015^b$. If you are incorrect you get -5 points; if you do not answer you get 0 points; otherwise you get $\max\{20 - 0.02(|b - a| - 1), 0\}$ points, rounded down to the nearest integer.

p35. Twin primes are prime numbers whose difference is 2. Let (a, b) be the 91717-th pair of twin primes, with $a < b$. Let $k = a^b$, and suppose that j is the number of digits in the base 10 representation of k. What is j^5? If the correct answer is n and you say m, you will receive $\max\left(20 - |\log\left(|\frac{m}{n}|\right), 0\right)$ points, rounded down to the nearest integer.

p36. Write down any positive integer. Let the sum of the valid submissions (i.e. positive integer submissions) for all teams be S. One team will be chosen randomly, according to the following distribution:
if your team's submission is n, you will be chosen with probability $\frac{n}{S}$. The amount of points that the chosen team will win is the greatest integer not exceeding $\min\{K, \frac{10000}{S}\}$. K is a predetermined secret value.

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