2017 Spring Lexington Math Tournament

www.artofproblemsolving.com/community/c2801171

by parmenides51

– Individual Round

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individual p1. Find the number of zeroes at the end of 20^{17}.

p2. Express $\frac{1}{\sqrt{20}+\sqrt{17}}$ in simplest radical form.

p3. John draws a square $ABCD$. On side AB he draws point P so that $\frac{BP}{PA}=\frac{1}{20}$ and on side BC he draws point Q such that $\frac{BQ}{QC}=\frac{1}{17}$. What is the ratio of the area of $\triangle PBQ$ to the area of $ABCD$?

p4. Alfred, Bill, Clara, David, and Emily are sitting in a row of five seats at a movie theater. Alfred and Bill don't want to sit next to each other, and David and Emily have to sit next to each other. How many arrangements can they sit in that satisfy these constraints?

p5. Alex is playing a game with an unfair coin which has a $\frac{1}{5}$ chance of flipping heads and a $\frac{4}{5}$ chance of flipping tails. He flips the coin three times and wins if he flipped at least one head and one tail. What is the probability that Alex wins?

p6. Positive two-digit number $\overline{ab}$ has 8 divisors. Find the number of divisors of the four-digit number $\overline{abab}$.

p7. Call a positive integer n diagonal if the number of diagonals of a convex n-gon is a multiple of the number of sides. Find the number of diagonal positive integers less than or equal to 2017.

p8. There are 4 houses on a street, with 2 on each side, and each house can be colored one of 5 different colors. Find the number of ways that the houses can be painted such that no two houses on the same side of the street are the same color and not all the houses are different colors.

p9. Compute

$$|2017-|2016|-|2015-|...|3-|2-1||...||||.$$

p10. Given points A,B in the coordinate plane, let $A\oplus B$ be the unique point C such that $\overline{AC}$

is parallel to the x-axis and $\overline{BC}$ is parallel to the y-axis. Find the point (x, y) such that $((x, y) \oplus (0, 1)) \oplus (1, 0) = (2016, 2017) \oplus (x, y)$.

p11. In the following subtraction problem, different letters represent different nonzero digits.

$$\begin{array}{cccc} & M & A & T & H \\ - & & H & A & M \\ \hline & & L & M & T \end{array}$$

How many ways can the letters be assigned values to satisfy the subtraction problem?

p12. If m and n are integers such that $17n + 20m = 2017$, then what is the minimum possible value of $|m - n|$?

p13. Let $f(x) = x^4 - 3x^3 + 2x^2 + 7x - 9$. For some complex numbers a, b, c, d, it is true that $f(x) = (x^2 + ax + b)(x^2 + cx + d)$ for all complex numbers x. Find $\frac{a}{b} + \frac{c}{d}$.

p14. A positive integer is called an imposter if it can be expressed in the form $2^a + 2^b$ where a, b are non-negative integers and $a \ne b$. How many almost positive integers less than 2017 are imposters?

p15. Evaluate the infinite sum

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{2^{n+1}} = \frac{1}{2} + \frac{3}{4} + \frac{6}{8} + \frac{10}{16} + \frac{15}{32} + ...$$

p16. Each face of a regular tetrahedron is colored either red, green, or blue, each with probability $\frac{1}{3}$. What is the probability that the tetrahedron can be placed with one face down on a table such that each of the three visible faces are either all the same color or all different colors?

p17. Let $(k, \sqrt{k})$ be the point on the graph of $y = \sqrt{x}$ that is closest to the point $(2017, 0)$. Find k.

p18. Alice is going to place 2016 rooks on a 2016×2016 chessboard where both the rows and columns are labelled 1 to 2016; the rooks are placed so that no two rooks are in the same row or the same column. The value of a square is the sum of its row number and column number. The score of an arrangement of rooks is the sumof the values of all the occupied squares. Find the average score over all valid configurations.

p19. Let $f(n)$ be a function defined recursively across the natural numbers such that $f(1) = 1$ and $f(n) = n^{f(n-1)}$. Find the sum of all positive divisors less than or equal to 15 of the number $f(7) - 1$.

p20. Find the number of ordered pairs of positive integers (m, n) that satisfy
$$gcd\,(m, n) + lcm\,(m, n) = 2017.$$

p21. Let $\triangle\ ABC$ be a triangle. Let M be the midpoint of AB and let P be the projection of A onto BC. If $AB = 20$, and $BC = MC = 17$, compute BP.

p22. For positive integers n, define the odd parent function, denoted $op(n)$, to be the greatest positive odd divisor of n. For example, $op(4) = 1$, $op(5) = 5$, and $op(6) = 3$. Find $\sum_{i=1}^{256} op(i)$.

p23. Suppose $\triangle\ ABC$ has sidelengths $AB = 20$ and $AC = 17$. Let X be a point inside $\triangle\ ABC$ such that $BX \perp CX$ and $AX \perp BC$. If $|BX^4 - CX^4| = 2017$, the compute the length of side BC.

p24. How many ways can some squares be colored black in a 6×6 grid of squares such that each row and each column contain exactly two colored squares? Rotations and reflections of the same coloring are considered distinct.

p25. Let $ABCD$ be a convex quadrilateral with $AB = BC = 2$, $AD = 4$, and $\angle ABC = 120^o$. Let M be the midpoint of BD. If $\angle AMC = 90^o$, find the length of segment CD.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Team Round

Team Round **p1.** Suppose that 20% of a number is 17. Find 20% of 17% of the number.

p2. Let A, B, C, D represent the numbers 1 through 4 in some order, with $A \neq 1$. Find the maximum possible value of $\frac{\log_A B}{C+D}$.

Here, $\log_A B$ is the unique real number X such that $A^X = B$.

p3. There are six points in a plane, no four of which are collinear. A line is formed connecting every pair of points. Find the smallest possible number of distinct lines formed.

p4. Let a, b, c be real numbers which satisfy
$$\frac{2017}{a} = a(b + c), \frac{2017}{b} = b(a + c), \frac{2017}{c} = c(a + b).$$

Find the sum of all possible values of abc.

p5. Let a and b be complex numbers such that $ab+a+b = (a+b+1)(a+b+3)$. Find all possible values of $\frac{a+1}{b+1}$.

p6. Let $\triangle ABC$ be a triangle. Let X, Y, Z be points on lines BC, CA, and AB, respectively, such that X lies on segment BC, B lies on segment AY , and C lies on segment AZ. Suppose that the circumcircle of $\triangle XYZ$ is tangent to lines AB, BC, and CA with center I_A. If $AB = 20$ and $I_AC = AC = 17$ then compute the length of segment BC.

p7. An ant makes 4034 moves on a coordinate plane, beginning at the point $(0, 0)$ and ending at $(2017, 2017)$. Each move consists of moving one unit in a direction parallel to one of the axes. Suppose that the ant stays within the region $|x - y| \le 2$. Let N be the number of paths the ant can take. Find the remainder when N is divided by 1000.

p8. A 10 digit positive integer $\overline{a_9a_8a_7...a_1a_0}$ with a_9 nonzero is called *deceptive* if there exist distinct indices $i > j$ such that $\overline{a_ia_j} = 37$. Find the number of deceptive positive integers.

p9. A circle passing through the points $(2, 0)$ and $(1, 7)$ is tangent to the y-axis at $(0, r)$. Find all possible values of r.

p10. An ellipse with major and minor axes 20 and 17, respectively, is inscribed in a square whose diagonals coincide with the axes of the ellipse. Find the area of the square.

PS. You had better use hide for answers.

Max Area The goal of this problem is to show that the maximum area of a polygon with a fixed number of sides and a fixed perimeter is achieved by a regular polygon.

(a) Prove that the polygon with maximum area must be convex. (Hint: If any angle is concave, show that the polygon's area can be increased.)

(b) Prove that if two adjacent sides have different lengths, the area of the polygon can be increased without changing the perimeter.

(c) Prove that the polygon with maximum area is equilateral, that is, has all the same side lengths.

It is true that when given all four side lengths in order of a quadrilateral, the maximum area is achieved in the unique configuration in which the quadrilateral is cyclic, that is, it can be inscribed in a circle.

(d) Prove that in an equilateral polygon, if any two adjacent angles are different then the area of the polygon
can be increased without changing the perimeter.

(e) Prove that the polygon of maximum area must be equiangular, or have all angles equal.

(f) Prove that the polygon of maximum area is a regular polygon.

PS. You had better use hide for answers.

Radical Cent Let P be a point and ω be a circle with center O and radius r . We define the power of the point P with respect to the circle ω to be $OP^2 - r^2$, and we denote this by pow (P, ω). We define the radical axis of two circles ω_1 and ω_2 to be the locus of all points P such that pow $(P, \omega_1) =$ pow (P, ω_2). It turns out that the pairwise radical axes of three circles are either concurrent or pairwise parallel. The concurrence point is referred to as the radical center of the three circles.

In $\triangle ABC$, let I be the incenter, Γ be the circumcircle, and O be the circumcenter. Let A_1, B_1, C_1 be the point of tangency of the incircle of $\triangle ABC$ with side BC, CA, AB, respectively. Let $X_1, X_2 \in \Gamma$ such that X_1, B_1, C_1, X_2 are collinear in this order. Let M_A be the midpoint of BC, and define ω_A as the circumcircle of $\triangle X_1X_2M_A$. Define ω_B ,ω_C analogously. The goal of this problem is to show that the radical center of ω_A, ω_B, ω_C lies on line OI.

(a) LetA_1' denote the intersection of B_1C_1 and BC. Show that $\frac{A_1B}{A_1C} = \frac{A_1'B}{A_1'C}$.
(b) Prove that A_1 lies on ω_A.
(c) Prove that A_1 lies on the radical axis of ω_B and ω_C .
(d) Prove that the radical axis of ω_B and ω_C is perpendicular to B_1C_1.
(e) Prove that the radical center of ω_A, ω_B, ω_C is the orthocenter of $\triangle A_1B_1C_1$.
(f) Conclude that the radical center of ω_A, ω_B, ω_C , O, and I are collinear.

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– Theme Round

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1 Building

– **p1**. Bob wants to build a bridge. This bridge has to be an arch bridge which reaches down on each side 10 feet and crosses a 20 foot gap. If the arch is shown by a semi circle with radius 9, and the bridge is 5 feet wide, how many cubic feet of material does Bob need?

p2. Evan constructs a ”poly-chain” by connecting regular polygons of side length 1 and having each adjacent polygon share a side. Additionally, Evan only creates a ”poly-chain” if the polygons in the chain all have a consecutive number of side lengths. For each ”poly-chain”, Evan then assigns it an ordered pair (m, n), where m is the number of polygons in the ”poly-chain”, and n

is the number of sides of the largest polygon. Find all ordered pairs (m, n) that correspond to a "poly-chain" with perimeter 17.

p3. Ben constructs a triangle ABC such that if M is the midpoint of BC then $AM = BC = 10$. Find the sum of all possible integer valued perimeters of $\triangle ABC$.

p4. Jason is creating a structure out of steel bars. First, he makes a cube. He then connects the midpoints of the faces to form a regular octahedron. He continues by connecting the midpoints of the faces of this octahedron to form another, smaller cube. Find the ratio of the volume of the smaller cube to the volume of the larger cube.

p5. Suppose Ben builds another triangle $\triangle ABC$ which has sidelengths $AB = 13$, $BC = 14$, $CA = 15$. Let D be the point of tangency between the incircle of $\triangle ABC$ and side BC, and let M be the midpoint of BC. The circumcircle of $\triangle ADM$ intersects the circumcircle of $\triangle ABC$ at a point $P \neq A$. If AP intersects BC at Q, find the length of BQ.

PS. You had better use hide for answers.

2 Music

– **p6.** Generally, only frequencies between 20 Hz and $20,000$ Hz are considered audible. Nathan has a special LMT clarinet that only plays notes at integer multiples of 2017 Hz. How many different audible notes can Nathan play on his clarinet?

p7. Nathan's chamber group of 6 people have to line up to take a photo. They have heights of $\{62, 65, 65, 67, 69, 70\}$. They must line up left to right with the rule that the heights of 2 people standing next to each other can differ by at most 3. Find the number of ways in which this chamber group can line up from left to right.

p8. Mark, who loves both music and math, plays middle A on his clarinet at a frequency A_0. Then, one by one, each one of his m students plays a note one octave above the previous one. Using his math skills, Mark finds that, rounding to the nearest tenth, $\log_2 A_0+\log_2 A_1+...+\log_2 A_m = 151.8$, where A_n denotes the frequency of the note n octaves above middle A. Given that $\log_2(A_0) = 8.8$, and that $A_n = A_0 \cdot 2^n$ for all positive integers n, how many students does Mark have?

p9. Janabel numbers the keys on her small piano from 1 to 10. She wants to choose a quintuplet of these keys (a, b, c), such that $a < b < c$ and each of these numbers are pairwise relatively prime. How many ways can she do this?

p10. Every day, John practices oboe in exact increments of either 0 minutes, 30 minutes, 1 hour, 1.5 hours, or 2 hours. How many possible ways can John practice oboe for a total of 5 hours in the span of 5 consecutive days?

PS. You had better use hide for answers.

3 Games

– **p11.** A deck of cards contains 4 suites with 13 numbers in each suit. Evan and Albert are playing a game with a deck of cards. First, Albert draws a card. Evan wins if he draws a card with the same number as Albert's card or the same suit as Albert's card. What is the probability that Evan wins?

p12. 2 squares in a square grid are called *adjacent* if they share a side. In the game of minesweeper, we have a 2017×2017 grid of squares such that each square adjacent to a square which contains a mine is marked (A square with a mine in it is not marked). Also, every square with a mine in it is adjacent to at least one square without a mine in it. Given that there are $5,000$ mines, what is the difference between the greatest and least number of marked squares?

p13. There are 2016 stones in a pile. Alfred and Bobby are playing a game where on each turn they can take either a or b stones from the pile where a and b are distinct integers less than or equal to 6. They alternate turns, with Albert going first, and the last person who is able to take a stone wins For example, if $a = 3$,$b = 6$ and after Alfred's turn there are 2 stones left, then Alfred wins because Bobby is unable to make a move. Let A represent the number of ordered pairs (a, b) for which Alfred has a winning strategy and B represent the number of ordered pairs (a, b) for which Bobby has a winning strategy. Find $A - B$.

p14. For a positive integer n define $f(n)$ to be the number of unordered triples of positive integers (a, b, c) such that
(a) $a, b, c \le n$, and
(b) There exists a triangle ABC with side lengths a, b, c and points D, E, F on line segments AB,BC,CA respectively such that AD,BE,CF all have integer side lengths and $ADEF$ is a parallelogram.
Evan and Albert play a game where they calculate $f(2017)$ and $f(2016)$. Find $f(2017) - f(2016)$.

p15. Two players A and B take turns placing counters in squares of an $1 \times n$ board, with A going first. Each turn, players must place a counter in a square does that not share an edge with any square that already has a counter in it. The first player who is unable to make a move loses. Find all $n \le 20$ for which A has a winning strategy.

PS. You had better use hide for answers.

– Guts Round

– Round 1

p1. Find all pairs (a, b) of positive integers with $a > b$ and $a^2 - b^2 = 111$.

p2. Alice drives at a constant rate of 2017 miles per hour. Find all positive values of x such that she can drive a distance of x^2 miles in a time of x minutes.

p3. ABC is a right triangle with right angle at B and altitude BH to hypotenuse AC. If $AB = 20$ and $BH = 12$, find the area of triangle $\triangle ABC$.

Round 2

p4. Regular polygons P_1 and P_2 have n_1 and n_2 sides and interior angles x_1 and x_2, respectively. If $\frac{n_1}{n_2} = \frac{7}{5}$ and $\frac{x_1}{x_2} = \frac{15}{14}$, find the ratio of the sum of the interior angles of P_1 to the sum of the interior angles of P_2.

p5. Joey starts out with a polynomial $f(x) = x^2 + x + 1$. Every turn, he either adds or subtracts 1 from f . What is the probability that after 2017 turns, f has a real root?

p6. Find the difference between the greatest and least positive integer values x such that $\sqrt[20]{\lfloor \sqrt[17]{x} \rfloor} = 1$.

Round 3

p7. Let $ABCD$ be a square and suppose P and Q are points on sides AB and CD respectively such that $\frac{AP}{PB} = \frac{20}{17}$ and $\frac{CQ}{QD} = \frac{17}{20}$. Suppose that $PQ = 1$. Find the area of square $ABCD$.

p8. If

$$\frac{\sum_{n\ge 0} r^n}{\sum_{n\ge 0} r^{2n}} = \frac{1 + r + r^2 + r^3 + ...}{1 + r^2 + r^4 + r^6 + ...} = \frac{20}{17},$$

find r .

p9. Let $\overline{abc}$ denote the 3 digit number with digits a, b and c. If $\overline{abc}_{10}$ is divisible by 9, what is the probability that $\overline{abc}_{40}$ is divisible by 9?

Round 4

p10. Find the number of factors of 20^{17} that are perfect cubes but not perfect squares.

p11. Find the sum of all positive integers $x \le 100$ such that x^2 leaves the same remainder as x does
upon division by 100.

p12. Find all b for which the base-b representation of 217 contains only ones and zeros.

PS. You should use hide for answers. Rounds 5-8 have been posted here (https://artofproblemsolving.com/community/c3h3158514p28715373).and 9-12 here (https://artofproblemsolving.com/community/c3h3162362p28764144) Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Round 5

p13. Two closed disks of radius $\sqrt{2}$ are drawn centered at the points $(1, 0)$ and $(-1, 0)$. Let P be the
region belonging to both disks. Two congruent non-intersecting open disks of radius r have all of
their points in P . Find the maximum possible value of r .

p14. A rectangle has positive integer side lengths. The sum of the numerical values of its perimeter and area is 2017. Find the perimeter of the rectangle.

p15. Find all ordered triples of real numbers (a, b, c) which satisfy

$$a + b + c = 6$$

$$a \cdot (b + c) = 6$$

$$(a + b) \cdot c = 6$$

Round 6

p16. A four digit positive integer is called confused if it is written using the digits 2, 0, 1, and 7 in some order, each exactly one. For example, the numbers 7210 and 2017 are confused. Find the sum of all confused numbers.

p17. Suppose $\triangle ABC$ is a right triangle with a right angle at A. Let D be a point on segment BC such that $\angle BAD = \angle CAD$. Suppose that $AB = 20$ and $AC = 17$. Compute AD.

p18. Let x be a real number. Find the minimum possible positive value of $\frac{|x-20|+|x-17|}{x}$.

Round 7

p19. Find the sum of all real numbers $0 < x < 1$ that satisfy $\{2017x\} = \{x\}$.

p20. Let $a_1, a_2, , , , , a_{10}$ be real numbers which sum to 20 and satisfy $\{a_i\} < 0.5$ for $1 \le i \le 10$. Find the sum of all possible values of $\sum_{1\le i<j\le 10} \lfloor a_i + a_j \rfloor$.

Here, $\lfloor x \rfloor$ denotes the greatest integer x_0 such that $x_0 \le x$ and $\{x\} = x - \lfloor x \rfloor$.

p21. Compute the remainder when 20^{2017} is divided by 17.

Round 8

p22. Let $\triangle ABC$ be a triangle with a right angle at B. Additionally, letM be the midpoint of AC. Suppose the circumcircle of $\triangle BCM$ intersects segment AB at a point $P \ne B$. If $CP = 20$ and $BP = 17$, compute AC.

p23. Two vertices on a cube are called neighbors if they are distinct endpoints of the same edge. On a cube, how many ways can a nonempty subset S of the vertices be chosen such that for any vertex $v \in S$, at least two of the three neighbors of v are also in S? Reflections and rotations are considered distinct.

p24. Let x be a real number such that $x + \sqrt[4]{5 - x^4} = 2$. Find all possible values of $x\sqrt[4]{5 - x^4}$.

PS. You should use hide for answers. Rounds 1-4 have been posted here (https://artofproblemsolving.com/community/c3h3158491p28715220).and 9-12 here (https://artofproblemsolving.com/community/c3h3162362p28764144) Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Round 9

p25. Let S be the set of the first 2017 positive integers. Find the number of elements $n \in S$ such that $\sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor$ is even.

p26. Let $\{x_n\}_{n\ge 0}$ be a sequence with $x_0 = 0$,$x_1 = \frac{1}{20}$,$x_2 = \frac{1}{17}$,$x_3 = \frac{1}{10}$, and $x_n = \frac{1}{2}((x_{n-2}+x_{n-4})$ for $n \ge 4$. Compute
$$\left\lfloor \frac{1}{x_{2017!} - x_{2017!-1}} \right\rfloor .$$

p27. Let $ABCDE$ be be a cyclic pentagon. Given that $\angle CEB = 17^o$, find $\angle CDE + \angle EAB$, in degrees.

Round 10

p28. Let $S = \{1, 2, 4, ..., 2^{2016}, 2^{2017}\}$. For each $0 \le i \le 2017$, let x_i be chosen uniformly at random from the subset of S consisting of the divisors of 2^i . What is the expected number of distinct values in the set $\{x_0, x_1, x_2, ..., x_{2016}, x_{2017}\}$?

p29. For positive real numbers a and b, the points $(a, 0)$, $(20, 17)$ and $(0, b)$ are collinear. Find the minimum possible value of $a + b$.

p30. Find the sum of the distinct prime factors of $2^{36} - 1$.

Round 11

p31. There exist two angle bisectors of the lines $y = 20x$ and $y = 17x$ with slopes m_1 and m_2. Find the unordered pair (m_1, m_2).

p32. Triangle 4ABC has sidelengths $AB = 13$, $BC = 14$, $CA = 15$ and orthocenter H. Let Ω_1 be the circle through B and H, tangent to BC, and let Ω_2 be the circle through C and H, tangent to BC. Finally, let $R \ne H$ denote the second intersection of Ω_1 and Ω_2. Find the length AR.

p33. For a positive integer n, let $S_n = \{1, 2, 3, ..., n\}$ be the set of positive integers less than or equal to n. Additionally, let
$$f(n) = |\{x \in S_n : x^{2017} \equiv x \ (mod\ n)\}|.$$

Find $f(2016) - f(2015) + f(2014) - f(2013)$.

Round 12

p34. Estimate the value of $\sum_{n=1}^{2017} \phi(n)$, where $\phi(n)$ is the number of numbers less than or equal n that are relatively prime to n. If your estimate is E and the correct answer is A, your score for this problem will be max $\max \left(0, \lfloor 15 - 75\frac{|A-E|}{A} \rceil \right)$.

p35. An up-down permutation of order n is a permutation σ of $(1, 2, 3, ..., n)$ such that $\sigma(i) < \sigma(i+1)$ if and only if i is odd. Denote by P_n the number of up-down permutations of order n. Estimate the value of $P_{20} + P_{17}$. If your estimate is E and the correct answer is A, your score for this problem will be $\max \left(0, 16 - \lceil \max \left(\frac{E}{A}, 2 - \frac{E}{A}\right) \rceil \right)$.

p36. For positive integers n, superfactorial of n, denoted $n\$$, is defined as the product of the first n factorials. In other words, we have $n\$ = \prod_{i=1}^{n}(i!)$. Estimate the number of digits in the product $(20\$) \cdot (17\$)$. If your estimate is E and the correct answer is A, your score for this problem will be $\max \left(0, \lfloor 15 - \frac{1}{2}|A-E| \rfloor \right)$.

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