Lexington Math Tournament

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by parmenides51

– Individual Round

Individual **p1.** Find the area of a right triangle with legs of lengths 20 and 18.

p2. How many 4-digit numbers (without leading zeros) contain only $2, 0, 1, 8$ as digits? Digits can be used more than once.

p3. A rectangle has perimeter 24. Compute the largest possible area of the rectangle.

p4. Find the smallest positive integer with 12 positive factors, including one and itself.

p5. Sammy can buy 3 pencils and 6 shoes for 9 dollars, and Ben can buy 4 pencils and 4 shoes for 10 dollars at the same store. How much more money does a pencil cost than a shoe?

p6. What is the radius of the circle inscribed in a right triangle with legs of length 3 and 4?

p7. Find the angle between the minute and hour hands of a clock at $12 : 30$.

p8. Three distinct numbers are selected at random fromthe set $\{1, 2, 3, ..., 101\}$. Find the probability that 20 and 18 are two of those numbers.

p9. If it takes 6 builders 4 days to build 6 houses, find the number of houses 8 builders can build in 9 days.

p10. A six sided die is rolled three times. Find the probability that each consecutive roll is less than the roll before it.

p11. Find the positive integer n so that $\frac{8-6\sqrt{n}}{n}$ is the reciprocal of $\frac{80+6\sqrt{n}}{n}$.

p12. Find the number of all positive integers less than 511 whose binary representations differ from that of 511 in exactly two places.

p13. Find the largest number of diagonals that can be drawn within a regular 2018-gon so that no two intersect.

p14. Let a and b be positive real numbers with $a > b$ such that $ab = a + b = 2018$. Find $\lfloor 1000a \rfloor$. Here $\lfloor x \rfloor$ is equal to the greatest integer less than or equal to x.

p15. Let r_1 and r_2 be the roots of $x^2 + 4x + 5 = 0$. Find $r_1^2 + r_2^2$.

p16. Let $\triangle ABC$ with $AB = 5$, $BC = 4$, $CA = 3$ be inscribed in a circle Ω. Let the tangent to Ω at A intersect BC at D and let the tangent to Ω at B intersect AC at E. Let AB intersect DE at F. Find the length BF.

p17. A standard 6-sided die and a 4-sided die numbered $1, 2, 3$, and 4 are rolled and summed. What is the probability that the sum is 5?

p18. Let A and B be the points $(2, 0)$ and $(4, 1)$ respectively. The point P is on the line $y = 2x+1$ such that $AP + BP$ is minimized. Find the coordinates of P.

p19. Rectangle $ABCD$ has points E and F on sides AB and BC, respectively. Given that $\frac{AE}{BE} = \frac{BF}{FC} = \frac12$, $\angle ADE = 30^o$, and $[DEF] = 25$, find the area of rectangle $ABCD$.

p20. Find the sum of the coefficients in the expansion of $(x^2 - x + 1)^{2018}$.

p21. If p, q and r are primes with $pqr = 19(p + q + r)$, find $p + q + r$.

p22. Let $\triangle ABC$ be the triangle such that $\angle B$ is acute and $AB < AC$. Let D be the foot of altitude from A to BC and F be the foot of altitude from E, the midpoint of BC, to AB. If $AD = 16$, $BD = 12$, $AF = 5$, find the value of AC^2.

p23. Let a, b, c be positive real numbers such that
(i) $c > a$
(ii) $10c = 7a + 4b + 2024$
(iii) $2024 = \frac{(a+c)^2}{a} + \frac{(c+a)^2}{b}$.
Find $a + b + c$.

p24. Let $f^1(x) = x^2 - 2x + 2$, and for $n > 1$ define $f^n(x) = f(f^{n-1}(x))$. Find the greatest prime factor of $f^{2018}(2019) - 1$.

p25. Let I be the incenter of $\triangle ABC$ and D be the intersection of line that passes through I that is perpendicular to AI and BC. If $AB = 60$, $CA = 120$, and $CD = 100$, find the length of BC.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Team Round

Team Round **p1.** Evaluate $1 + 3 + 5 + \quad + 2019$.

p2. Evaluate $1^2 - 2^2 + 3^2 - 4^2 + ... + 99^2 - 100^2$.

p3. Find the sum of all solutions to $|2018 + |x - 2018|| = 2018$.

p4. The angles in a triangle form a geometric series with common ratio $\frac12$. Find the smallest angle in the triangle.

p5. Compute the number of ordered pairs (a, b, c, d) of positive integers $1 \le a, b, c, d \le 6$ such that $ab + cd$ is a multiple of seven.

p6. How many ways are there to arrange three birch trees, four maple, and five oak trees in a row if trees of the same species are considered indistinguishable.

p7. How many ways are there for Mr. Paul to climb a flight of 9 stairs, taking steps of either two or three at a time?

p8. Find the largest natural number x for which x^x divides $17!$

p9. How many positive integers less than or equal to 2018 have an odd number of factors?

p10. Square $MAIL$ and equilateral triangle LIT share side IL and point T is on the interior of the square. What is the measure of angle LMT?

p11. The product of all divisors of 2018^3 can be written in the form $2^a \cdot 2018^b$ for positive integers

a and b. Find $a + b$.

p12. Find the sum all four digit palindromes. (A number is said to be palindromic if its digits read the same forwards and backwards.

p13. How ways are there for an ant to travel from point $(0, 0)$ to $(5, 5)$ in the coordinate plane if it may only move one unit in the positive x or y directions each step, and may not pass through the point $(1, 1)$ or $(4, 4)$?

p14. A certain square has area 6. A triangle is constructed such that each vertex is a point on the perimeter of the square. What is the maximum possible area of the triangle?

p15. Find the value of ab if positive integers a, b satisfy $9a^2 - 12ab + 2b^2 + 36b = 162$.

p16. $\triangle ABC$ is an equilateral triangle with side length 3. Point D lies on the segment BC such that $BD = 1$ and E lies on AC such that $AE = AD$. Compute the area of $\triangle ADE$.

p17. Let $A_1, A_2, ..., A_{10}$ be 10 points evenly spaced out on a line, in that order. Points B_1 and B_2 lie on opposite sides of the perpendicular bisector of A_1A_{10} and are equidistant to l. Lines $B_1A_1, ..., B_1A_{10}$ and $B_2A_1, ..., B_2A_{10}$ are drawn. How many triangles of any size are present?

p18. Let $T_n = 1+2+3+n$ be the nth triangular number. Determine the value of the infinite sum $\sum_{k\ge1} \frac{T_k}{2^k}$.

p19. An infinitely large bag of coins is such that for every $0.5 < p \le 1$, there is exactly one coin in the bag with probability p of landing on heads and probability $1 - p$ of landing on tails. There are no other coins besides these in the bag. A coin is pulled out of the bag at random and when flipped lands on heads. Find the probability that the coin lands on heads when flipped again.

p20. The sequence $\{x_n\}_{n\ge1}$ satisfies $x1 = 1$ and $(4 + x_1 + x_2 + \ + x_n)(x_1 + x_2 + \ + x_{n+1}) = 1$ for all $n \ge 1$. Compute $\left\lfloor \frac{x_{2018}}{x_{2019}} \right\rfloor$.

PS. You had better use hide for answers.

–	Theme Round
1	Chemistry

– Chemistry. BOOM BOOM. Things explode, and if you're not careful, you might explode, too.

p1. The half life of a radioactive isotope is the time it takes for the isotope to decay too half of its original concentration. Francium-223 decays with a half life of 22 minutes. Determine how many minutes it takes for a sample of francium-223 to decay to 25% of its original concentration.

p2. Ezra is walking across the periodic table. Determine the number of ways he can walk from gallium, atomic symbol Ga, to neon, atomic symbol Ne, if he must walk through sulfur, atomic symbol S, and can only walk right or up.

https://cdn.artofproblemsolving.com/attachments/1/6/ff219ad16c7a082e01d40ce13a6577bdcd01e
png

p3. In organic chemistry, molecules can be represented as polygons, where vertices are atoms and edges are bonds. A certain molecule is a finite plane of continuous hexagons, such as the one shown below. If there are 32 atoms and 41 bonds, how many hexagons are in the molecule?

https://cdn.artofproblemsolving.com/attachments/a/e/fd3f1a2a1cc9ad19c0440a160d678a7730e16
png

p4. Steel is an alloy of iron and carbon. Four iron atoms can be represented as mutually tangent spheres, and the carbon atom can be represented as a sphere externally tangent to all four iron atoms. If the radius of the iron atom is 12 angstroms, determine the radius of a carbon atom in angstroms.

p5. Electrons in an atom are described by a set of four quantum numbers, $\{n, l, m_l, m_s\}$, according to the following restrictions: all quantum numbers must be integers except m_s, which can be either $+\frac{1}{2}$ or $-\frac{1}{2}$, $0 \le l < n$ and $|m_l| \le l$. Additionally, no two electrons can have the same set of four quantum numbers. Determine the number of electrons that can exist such that $n \le 20$.

PS. You should use hide for answers.

2 Mafia

– Mafia is a game where there are two sides: The village and the Mafia. Every night, the Mafia kills a person who is sided with the village. Every day, the village tries to hunt down the Mafia through communication, and at the end of every day, they vote on who they think the mafia are.

p6. Patrick wants to play a game of mafia with his friends. If he has 10 friends that might show up to play, each with probability $1/2$, and they need at least 5 players and a narrator to play, what is the probability that Patrick can play?

p7. At least one of Kathy and Alex is always mafia. If there are 2 mafia in a game with 6 players, what is the probability that both Kathy and Alex are mafia?

p8. Eric will play as mafia regardless of whether he is randomly selected to be mafia or not, and Euhan will play as the town regardless of what role he is selected as. If there are 2 mafia and 6 town, what is the expected value of the number of people playing as mafia in a random game with Eric and Euhan?

p9. Ben is trying to cheat in mafia. As a mafia, he is trying to bribe his friend to help him win the game with his spare change. His friend will only help him if the change he has can be used to form at least 25 different values. What is the fewest number of coins he can have to achieve this added to the fewest possible total value of those coins? He can only use pennies, nickels, dimes, and quarters.

p10. Sammy, being the very poor mafia player he is, randomly shoots another player whenever he plays as the vigilante. What is the probability that the player he shoots is also not shot by the mafia nor saved by the doctor, if they both select randomly in a game with 8 people? There are 2 mafia, and they cannot select a mafia to be killed, and the doctor can save anyone.

PS. You should use hide for answers.

3 Music

– What music do you like the best? Perhaps theme music from anime? Or maybe you like gaming theme music from YouTubing gamers on arcade games!

p11. Every note in a musical phrase that is 2 measures long lasts for $1/8$ of the measure of $1/4$ of it. How many different rhythms can the phrase have if there are no rests?

p12. I play a random chord of three different notes. It is called dissonant if the distance between any two of those notes are 1, 2, 6, 10, or 11 semitones apart. Given that the longest distance between two notes is at most 12 semitones, what is the probability that the chord will be dissonant?

p13. Two distinct notes are called an interval and if they are $0, 5$, or 7 semitones apart modulo 12, they are a perfect interval. (Modulo 12 means the remainder when divided by 12.) If a piano has 88 notes, and consecutive notes are 1 semitone apart, how many perfect intervals can be played?

p14. An ensemble has a violin, a viola, and a cello. A chord has 3 notes and each instrument can play 1 or 2 of them at a time. How many ways can they play the chord if every note in the chord must be played? (Octaves don't matter.)

PS. You should use hide for answers.

– Guts Round

– Round 1

p1. Evaluate the sum $1-2+3-...-208+209-210$.

p2. Tony has 14 beige socks, 15 blue socks, 6 brown socks, 8 blond socks and 7 black socks. If Tony picks socks out randomly, how many socks does he have to pick in order to guarantee a pair of blue socks?

p3. The price of an item is increased by 25%, followed by an additional increase of 20%. What is the overall percentage increase?

Round 2

p4. A lamp post is 20 feet high. How many feet away from the base of the post should a person who is 5 feet tall stand in order to cast an 8-foot shadow?

p5. How many positive even two-digit integers are there that do not contain the digits 0, 1, 2, 3 or 4?

p6. In four years, Jack will be twice as old as Jill. Three years ago, Jack was three times as old as Jill. How old is Jack?

Round 3

p7. Let $x\Delta y = xy^2-2y$. Compute $20\Delta 18$.

p8. A spider crawls 14 feet up a wall. If Cheenu is standing 6 feet from the wall, and is 6 feet tall, how far must the spider jump to land on his head?

p9. There are fourteen dogs with long nails and twenty dogs with long fur. If there are thirty dogs in total, and three do not have long fur or long nails, how many dogs have both long hair and long nails?

Round 4

p10. Exactly 420 non-overlapping square tiles, each 1 inch by 1 inch, tesselate a rectangle. What is the least possible number of inches in the perimeter of the rectangle?

p11. John drives 100 miles at fifty miles per hour to see a cat. After he discovers that there was no cat, he drives back at a speed of twenty miles per hour. What was John's average speed in the round trip?

p12. What percent of the numbers $1, 2, 3, ..., 1000$ are divisible by exactly one of the numbers 4 and 5?

PS. You should use hide for answers. Rounds 5-8 have been posted here (`https://artofproblemsolving.com/community/c3h3165992p28809294`) and 9-12 here (`https://artofproblemsolving.com/community/c3h3166045p28809814`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 5

p13. Express the number 3024_8 in base 2.

p14. $\triangle ABC$ has a perimeter of 10 and has $AB = 3$ and $\angle C$ has a measure of 60^o. What is the maximum area of the triangle?

p15. A weighted coin comes up as heads 30% of the time and tails 70% of the time. If I flip the coin 25 times, howmany tails am I expected to flip?

Round 6

p16. A rectangular box with side lengths 7, 11, and 13 is lined with reflective mirrors, and has edges aligned with the coordinate axes. A laser is shot from a corner of the box in the direction of the line $x = y = z$. Find the distance traveled by the laser before hitting a corner of the box.

p17. The largest solution to $x^2 + \frac{49}{x^2} = 2018$ can be represented in the form $\sqrt{a} + \sqrt{b}$. Compute $a + b$.

p18. What is the expected number of black cards between the two jokers of a 54 card deck?

Round 7

p19. Compute $\binom{6}{0} \cdot 2^0 + \binom{6}{1} \cdot 2^1 + \binom{6}{2} \cdot 2^2 + ... + \binom{6}{6} \cdot 2^6$.

p20. Define a sequence by $a_1 = 5$, $a_{n+1} = a_n + 4 * n - 1$ for $n \ge 1$. What is the value of a_{1000}?

p21. Let $\vartriangle ABC$ be the triangle such that $\angle B = 15^o$ and $\angle C = 30^o$. Let D be the point such that $\vartriangle ADC$ is an isosceles right triangle where D is in the opposite side from A respect to BC and $\angle DAC = 90^o$. Find the $\angle ADB$.

Round 8

p22. Say the answer to problem 24 is z. Compute $gcd(z, 7z + 24)$.

p23. Say the answer to problem 22 is x. If x is 1, write down 1 for this question. Otherwise, compute

$$\sum_{k=1}^{\infty} \frac{1}{x^k}$$

p24. Say the answer to problem 23 is y. Compute

$$\left\lfloor \frac{y^2 + 1}{y} \right\rfloor$$

PS. You should use hide for answers. Rounds 1-4 have been posted here (https://artofproblemsolving.com/community/c3h3165983p28809209) and 9-12 here (https://artofproblemsolving.com/community/c3h3166045p28809814). Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Round 9

p25. A positive integer is called spicy if it is divisible by the sum if its digits. Find the number of spicy integers between 100 and 200 inclusive.

p26. Rectangle $ABCD$ has points E and F on sides AB and BC, respectively. Given that $\frac{AE}{BE} = \frac{BF}{FC} = \frac{1}{2}$, $\angle ADE = 30^o$, and $[DEF] = 25$, find the area of rectangle $ABCD$.

p27. Find the largest value of n for which 3^n divides $\binom{100}{33}$.

Round 10

p28. Isosceles trapezoid $ABCD$ is inscribed in a circle such that $AB \parallel CD$, $AB = 2$, $CD = 4$, and $AC = 9$. What is the radius of the circle?

p29. Find the product of all possible positive integers n less than 11 such that in a group of n people, it is possible for every person to be friends with exactly 3 other people within the group. Assume that friendship is amutual relationship.

p30. Compute the infinite product

$$\left(1+\frac{1}{2^1}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{2^4}\right)\left(1+\frac{1}{2^8}\right)\left(1+\frac{1}{2^{16}}\right)...$$

Round 11

p31. Find the sum of all possible values of xy if $x+\frac{1}{y} = 12$ and $\frac{1}{x}+y = 8$.

p32. Find the number of ordered pairs (a, b), where $0 < a, b < 1999$, that satisfy $a^2 + b^2 \equiv ab$ (mod 1999)

p33. Let $f : N \to Q$ be a function such that $f(1) = 0$, $f(2) = 1$ and $f(n) = \frac{f(n-1)+f(n-2)}{2}$. Evaluate

$$\lim_{n\to\infty} f(n).$$

Round 12

p34. Estimate the sumof the digits of 2018^{2018}. The number of points you will receive is calculated using the formula $\max(0, 15 - \log_{10}(A - E))$, where A is the true value and E is your estimate.

p35. Let $C(m, n)$ denote the number of ways to tile an m by n rectangle with 1×2 tiles. Estimate $\log_{10}(C(100, 2))$. The number of points you will recieve is calculated using the formula $\max(0, 15 - \log_{10}(A - E))$, where A is the true value and E is your estimate.

p36. Estimate $\log_2 \binom{1000}{500}$. The number of points you earn is equal to $\max(0, 15 - |A - E|)$, where A is the true value and E is your estimate.

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