Lexington Math Tournament
www.artofproblemsolving.com/community/c2812349
by parmenides51

– Individual Round

Individual **p1.** Evaluate $6^4 + 5^4 + 3^4 + 2^4$.

p2. What digit is most frequent between 1 and 1000 inclusive?

p3. Let $n = gcd\left(2^2 \cdot 3^3 \cdot 4^4, 2^4 \cdot 3^3 \cdot 4^2\right)$. Find the number of positive integer factors of n.

p4. Suppose p and q are prime numbers such that $13p + 5q = 91$. Find $p + q$.

p5. Let $x = (5^3 - 5)(4^3 - 4)(3^3 - 3)(2^3 - 2)(1^3 - 1)$. Evaluate 2018^x .

p6. Liszt the lister lists all 24 four-digit integers that contain each of the digits $1, 2, 3, 4$ exactly once in increasing order. What is the sum of the 20th and 18th numbers on Liszt's list?

p7. Square $ABCD$ has center O. Suppose M is the midpoint of AB and $OM + 1 = OA$. Find the area of square $ABCD$.

p8. How many positive 4-digit integers have at most 3 distinct digits?

p9. Find the sumof all distinct integers obtained by placing $+$ and $-$ signs in the following spaces

$$2\_3\_4\_5$$

p10. In triangle ABC, $\angle A = 2\angle B$. Let I be the intersection of the angle bisectors of B and C. Given that $AB = 12$, $BC = 14$,and $CA = 9$, find AI .

p11. You have a $3 \times 3 \times 3$ cube in front of you. You are given a knife to cut the cube and you are allowed to move the pieces after each cut before cutting it again. What is the minimumnumber of cuts you need tomake in order to cut the cube into 27 $1 \times 1 \times 1$ cubes?

p12. How many ways can you choose 3 distinct numbers fromthe set $\{1, 2, 3, ..., 20\}$ to create a geometric sequence?

p13. Find the sum of all multiples of 12 that are less than 10^4 and contain only 0 and 4 as digits.

p14. What is the smallest positive integer that has a different number of digits in each base from 2 to 5?

p15. Given 3 real numbers (a, b, c) such that
$$\frac{a}{b+c}=\frac{b}{3a+3c}=\frac{c}{a+3b},$$
find all possible values of $\frac{a+b}{c}$.

p16. Let S be the set of lattice points (x, y, z) in R^3 satisfying $0 \le x, y, z \le 2$. How many distinct triangles exist with all three vertices in S?

p17. Let $\oplus$ be an operator such that for any 2 real numbers a and b, $a \oplus b = 20ab - 4a - 4b + 1$. Evaluate
$$\frac{1}{10}\oplus\frac{1}{9}\oplus\frac{1}{8}\oplus\frac{1}{7}\oplus\frac{1}{6}\oplus\frac{1}{5}\oplus\frac{1}{4}\oplus\frac{1}{3}\oplus\frac{1}{2}\oplus 1.$$

p18. A function $f : N \to N$ satisfies $f(f(x)) = x$ and $f(2f(2x + 16)) = f\left(\frac{1}{x+8}\right)$ for all positive integers x. Find $f(2018)$.

p19. There exists an integer divisor d of 240100490001 such that $490000 < d < 491000$. Find d.

p20. Let a and b be not necessarily distinct positive integers chosen independently and uniformly at random from the set $\{1, 2, 3, ..., 511, 512\}$. Let $x = \frac{a}{b}$. Find the probability that $(-1)^x$ is a real number.

p21. In $\vartriangle ABC$ we have $AB = 4$, $BC = 6$, and $\angle ABC = 135^o$. $\angle ABC$ is trisected by rays B_1 and B_2. Ray B_1 intersects side CA at point F, and ray B_2 intersects side CA at point G. What is the area of $\vartriangle BFG$?

p22. A level number is a number which can be expressed as $x \cdot \lfloor x\rfloor \cdot \lceil x\rceil$ where x is a real number. Find the number of positive integers less than or equal to 1000 which are also level numbers.

p23. Triangle $\vartriangle ABC$ has sidelengths $AB = 13$, $BC = 14$, $CA = 15$ and circumcenter O. Let D be the intersection of AO and BC. Compute BD/DC.

p24. Let $f(x) = x^4 - 3x^3 + 2x^2 + 5x - 4$ be a quartic polynomial with roots a, b, c, d. Compute

$$\left(a+1+\frac{1}{a}\right)\left(b+1+\frac{1}{b}\right)\left(c+1+\frac{1}{c}\right)\left(d+1+\frac{1}{d}\right).$$

p25. Triangle $\triangle ABC$ has centroid G and circumcenter O. Let D be the foot of the altitude from A to BC. If $AD = 2018$, $BD = 20$, and $CD = 18$, find the area of triangle $\triangle DOG$.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Team Round

Team Round p1. Points $P_1, P_2, P_3, ..., P_n$ lie on a plane such that $P_aP_b = 1$,$P_cP_d = 2$, and $P_eP_f = 2018$ for not necessarily distinct indices $a, b, c, d, e, f \in \{1, 2, ..., n\}$. Find the minimum possible value of n.

p2. Find the coefficient of the x^2y^4 term in the expansion of $(3x + 2y)^6$.

p3. Find the number of positive integers $n < 1000$ such that n is a multiple of 27 and the digit sum of n is a multiple of 11.

p4. How many times do the minute hand and hour hand of a 12-hour analog clock overlap in a 366-day leap year?

p5. Find the number of ordered triples of integers (a, b, c) such that $(a + b)(b + c)(c + a) = 2018$.

p6. Let S denote the set of the first 2018 positive integers. Call the score of a subset the sum of its maximal element and its minimal element. Find the sum of score (x) over all subsets $s \in S$

p7. How many ordered pairs of integers (a, b) exist such that $1 \le a, b \le 20$ and a^a divides b^b?

p8. Let f be a function such that for every non-negative integer p, $f(p)$ equals the number of ordered pairs of positive integers (a, n) such that $a^n = a^p \cdot n$. Find $\sum_{p=0}^{2018} f(p)$.

p9. A point P is randomly chosen inside a regular octagon $A_1A_2A_3A_4A_5A_6A_7A_8$. What is the probability that the projections of P onto the lines $\overleftrightarrow{A_iA_{i+1}}$ for $i = 1, 2, ..., 8$ lie on the segments $\overline{A_iA_{i+1}}$ for $i = 1, 2, ..., 8$ (where indices are taken $mod\ 8$)?

p10. A person keeps flipping an unfair coin until it flips 3 tails in a row. The probability of it landing on heads is $\frac{2}{3}$ and the probability it lands on tails is $\frac{1}{3}$. What is the expected value of the number of the times the coin flips?

PS. You had better use hide for answers.

–	Theme Round
1	Olympics

– **p1.** The number of ways to rearrange PYEONGCHANG OLYMPIC can be expressed as $\frac{L!}{(M!)^T}$. Compute $L + M + T$.

p2. How many ways are there to color each ring in the olympic logo with the colors blue, yellow, black, green or red such that no two rings that intersect are the same color?
https://cdn.artofproblemsolving.com/attachments/0/b/e32579790b5d633419a86d0c15af4ccd69849
png

p3. At the Winter Olympics, Belarus and the Czech Republic won B and C medals respectively. Altogether, they won 10 medals. When the number of medals the US won is divided by $B + C$ it leaves a remainder of 3 and when it is divided by BC it leaves a remainder of 2. What is the least possible number of medals the U.S. could have won?

p4. Let ℓ_1 denote the string CURLING. Let ℓ_{k+1} be the sequence formed by inserting the substring 'CURLING' between each consecutive letter of lk . For example,

$$\ell_2 = \mathbf{C}CURLING\mathbf{U}CURLING\mathbf{R}CURLING\mathbf{L}CURLING\mathbf{I}CURLING\mathbf{N}CURLING\mathbf{G}.$$

What is the 1537th letter in ℓ_{2018}?

p5. Five spectators of an Olympic wrestling match each stand at a random point around the circumference of the circular ring. Find the probability that they are all contained within a 90 degree arc of the circle.

PS. You should use hide for answers.

2	Trivia

– **p6.** Randy is playing a trivia game with 6 questions. Each question has 3 answer choices and if he answers all 6 questions correctly, he wins 5000 dollars. What is the expected amount of money Randy will win?

p7. It has recently been proven that a sudoku puzzle requires at least 17 numbers to be uniquely solvable. We section a 4×4 grid of boxes into four 2×2 squares. In each square we place the number $1, 2, 3$, or 4. An arrangement is called *sudoku-like* if there is exactly one of $1, 2, 3, 4$ in each row, column, and 2×2 box. How many sudoku-like arrangements are there? An example is given below.
`https://cdn.artofproblemsolving.com/attachments/6/a/f3c94ba7f064932c3789ac31880e2170b9a4e`
`png`

p8. A *superperfect* number is a number n such that if $\sigma(n)$ denotes the sum of its factors, then $\sigma(\sigma(n)) = 2n$. For example, $\sigma(16) = 1+2+4+8+16 = 31$ and $\sigma(31) = 32$. A and B are distinct numbers such that they each have exactly 10 factors and $|A-B| = 1$. Find the minimum possible value of $A+B$.

p9. A trivia game awards up to 2018 dollars split evenly among all of its winners such that each winner gets the maximum possible integer number of dollars. In a particular game if one more person had won each winner would have gotten two fewer dollars. How many possible number of winners are there for this game?

p10. The number 0 didn't exist until 628 AD, when it was introduced by the Indian mathematician Brahmagupta. The concept of 0 did exist much before then.

Let $\oplus$ be a binary operator such that for any 3 real numbers a, b, c, we have

$$(a\oplus b)\oplus c = a\oplus (b\oplus c)$$

and

$$a\oplus b\oplus c = 4abc - 2ab - 2bc - 2ac + a + b + c.$$

Find all possible values of $20\oplus 18$.

PS. You should use hide for answers.

3 Star Wars

– **p11.** R2-D2 is trying to break into a room. He realizes that the code is the same as $A+B$, where A and B are two positive integers greater than 1 such that

$$20_A + 12_B = 19_{B+A}$$

and

$$20_A - 12_B = 11_{B-A},$$

where the A subscript means the number is in base A. What is the password?

p12. Tatooine is located at the point $(0, 4)$, Hoth is located at the point $(3, 5)$, Alderaan was located at the point $(2, 1)$ and Endor is located at $(0, 0)$. At what point should the Death Star stop in order to minimize its combined distance to each location?

p13. How many ways can Chancellor Palpatine place indistinguishable galactic credits on a 4 by 8 grid such that each row has exactly 3 credits and each column has exactly 6 credits?

p14. Rey and Kylo Ren are each flying through space. Rey starts at $(-2, 2)$ and ends at $(2, 2)$ and Kylo Ren starts at $(-1, 1)$ and ends at $(1, 1)$. They can only travel 1 unit in the x or y direction at a time. Additionally, each of them must go to the x-axis before going to their respective destinations and take the shortest path possible to do so. Finally, their paths never intersect at any point. How many distinct configurations of paths can they take?

p15. A triangle ABC is formed with A is the current location of Darth Vader's spaceship, B is the location of the Rebel Base, and C is the location of the Death Star. Let D, E, and F be the locations of the spaceships of Luke Skywalker, Han Solo, and Princess Leia, respectively. It is true that D is the foot of the A-altitude, E is the foot of the A-angle bisector, and F is the foot of the A-median. Suppose the 4 segments on BC (some possibly of length 0), when measured in light-years, form an arithmetic sequence (in any order). What is the largest possible value of $\frac{BC}{AB}$?

PS. You should use hide for answers.