Lexington Math Tournament

www.artofproblemsolving.com/community/c2812358

by parmenides51

– Team Round

Team Round p1. David runs at 3 times the speed of Alice. If Alice runs 2 miles in 30 minutes, determine how many minutes it takes for David to run a mile.

p2. Al has 2019 red jelly beans. Bob has 2018 green jelly beans. Carl has x blue jelly beans. The minimum number of jelly beans that must be drawn in order to guarantee 2 jelly beans of each color is 4041. Compute x.

p3. Find the 7-digit palindrome which is divisible by 7 and whose first three digits are all 2.

p4. Determine the number of ways to put 5 indistinguishable balls in 6 distinguishable boxes.

p5. A certain reduced fraction $\frac{a}{b}$ (with $a, b > 1$) has the property that when 2 is subtracted from the numerator and added to the denominator, the resulting fraction has $\frac{1}{6}$ of its original value. Find this fraction.

p6. Find the smallest positive integer n such that $|\tau(n + 1) - \tau(n)| = 7$. Here, $\tau(n)$ denotes the number of divisors of n.

p7. Let $\vartriangle ABC$ be the triangle such that $AB = 3$, $AC = 6$ and $\angle BAC = 120^o$. Let D be the point on BC such that AD bisect $\angle BAC$. Compute the length of AD.

p8. 26 points are evenly spaced around a circle and are labeled A through Z in alphabetical order. Triangle $\vartriangle LMT$ is drawn. Three more points, each distinct from L, M, and T , are chosen to form a second triangle. Compute the probability that the two triangles do not overlap.

p9. Given the three equations $a + b + c = 0$ $a^2 + b^2 + c^2 = 2$ $a^3 + b^3 + c^3 = 19$
find abc.

p10. Circle ω is inscribed in convex quadrilateral $ABCD$ and tangent to AB and CD at P and Q, respectively. Given that $AP = 175$, $BP = 147$,$CQ = 75$, and $AB \parallel CD$, find the length of DQ.

p11. Let p be a prime and m be a positive integer such that $157p = m^4 + 2m^3 + m^2 + 3$. Find the ordered pair (p, m).

p12. Find the number of possible functions $f : \{-2, -1, 0, 1, 2\} \to \{-2, -1, 0, 1, 2\}$ that satisfy the following conditions.
(1) $f(x) \ne f(y)$ when $x \ne y$
(2) There exists some x such that $f(x)^2 = x^2$

p13. Let p be a prime number such that there exists positive integer n such that $41pn - 42p^2 = n^3$. Find the sum of all possible values of p.

p14. An equilateral triangle with side length 1 is rotated 60 degrees around its center. Compute the area of the region swept out by the interior of the triangle.

p15. Let $\sigma(n)$ denote the number of positive integer divisors of n. Find the sum of all n that satisfy the equation $\sigma(n) = \frac{n}{3}$.

p16. Let C be the set of points $\{a, b, c\} \in Z$ for $0 \le a, b, c \le 10$. Alice starts at $(0, 0, 0)$. Every second she randomly moves to one of the other points in C that is on one of the lines parallel to the x, y, and z axes through the point she is currently at, each point with equal probability. Determine the expected number of seconds it will take her to reach $(10, 10, 10)$.

p17. Find the maximum possible value of

$$abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^3$$

where a, b, c are real such that $a + b + c = 0$.

p18. Circle ω with radius 6 is inscribed within quadrilateral $ABCD$. ω is tangent to AB, BC, CD, and DA at E, F, G, and H respectively. If $AE = 3$, $BF = 4$ and $CG = 5$, find the length of DH.

p19. Find the maximum integer p less than 1000 for which there exists a positive integer q such that the cubic equation

$$x^3 - px^2 + qx - (p^2 - 4q + 4) = 0$$

has three roots which are all positive integers.

p20. Let $\triangle ABC$ be the triangle such that $\angle ABC = 60^o$,$\angle ACB = 20^o$. Let P be the point such that CP bisects $\angle ACB$ and $\angle PAC = 30^o$. Find $\angle PBC$.

PS. You had better use hide for answers.

– Theme Round

1 Pick Up Lines

– **p1.** Anka wants to know if it hurt when her crush fell from the sky. She curls up into a ball and jumps off a 80 meter high building. If she bounces up to $3/4$ of the previous height each bounce, how many times can she bounce while still moving a total distance of less than 300 meters?

p2. Alex wants to rearrange the alphabet to put him and his crush next to each other. If he randomly rearranges it, what is the probability that ”u” and ”i” are next to each other?

p3. Jeffrey, being from Tennessee, sees 10s everywhere he looks. If he assigns to each of his 10000 lovers a unique integer ID number from 1 to 10000 how many of them will have the sequence ”10” in their ID?

p4. Andrew is getting lost in Amy's eyes, or more specifically, her i's: $a_i, b_i, c_i, ..., z_i$.
Let $a_n = \frac{1}{2^n}$,$b_n = \frac{1}{3^n}$, ..., $z_n = \frac{1}{27^n}$.
Additionally, let $S = \{(i_1, i_2, ..., i_{26})|i_1 \ge 1, i_2, i_3, ..., i_{26} \ge 0, i_1, i_2, ..., i_{26} \in Z\}$.
Find the sum of $a_{i_1} b_{i_2} ... z_{i_{26}}$ over all $(i_1, i_2, ..., i_{26}) \in S$.

p5. Janabel is in love with regular pentagon $ANGEL$ with side length 4 and area x. Find x^2.

PS. You should use hide for answers.

2 Goldilocks Act 1

– **p6.** Goldilocks is walking around in the magical forked forest. She has a $1/2$ chance of choosing the correct path at each fork. If she chooses the wrong path more than two times, she gets lost. What is the probability she gets lost if she encounters five forks?

p7. Goldilocks is being followed by a squirrel. Whenever the squirrel is 20 ft behind Goldilocks, it runs up until it catches up with her, then stays in place until Goldilocks is 20 ft ahead again. The squirrel runs at a rate of 10 ft/s and Goldilocks walks at a rate of 5 ft/s. If Goldilocks and the squirrel start at the same place, how many seconds will pass before the squirrel catches up to Goldilocks again?

p8. Goldilocks walks up to the door, and observes that there are ten locks. She looks under the mattress and finds ten keys. To open the door, every key must be in the correct lock. Moreover, if she puts one key in its correct lock, it cannot be removed anymore. If she tries a random combination of the remaining keys every minute, what is the expected number of minutes until she opens the door?

p9. A river runs parallel to Goldilocks' trail. Two congruent circular fields are tangent to the trail, the river, and each other. A third smaller field is externally tangent to the two fields and the river. If the distance between the river and the trail is 200 feet, what is the radius of the smaller field?

p10. Goldilocks stumbles upon a large house with an even larger field of flowers. She reads a sign that says, "Three bears in the house, 3^{2019} flowers in the field." If the flowers are arranged in rows of 100, how many flowers will be left over?

PS. You should use hide for answers.

3 Goldilocks Act 2

– **p11.** As the bears walk back from their home, they realize they left their fish by the river. The river is a circle centered at their home with radius 6 miles. If they are currently 4 miles away from their home, what is the shortest possible distance, in miles, that they must travel to get to the river and then return home?

p12. Goldilocks walks into the bedroom and sees three beds, B_1, B_2, and B_3. Sleepy from her meal, she decides to take a nap in each bed. She slept three times as long in B_2 as B_1 and ten minutes less in B_1 as B_3. If the most time she slept in any single bed was 2019 minutes, how long did she sleep in total?

p13. The Bears are on their way back from a stroll. They will return home at a random time between 3 and 4 pm, while Goldilocks will wake up at a random time between 2 and 4 pm. The bears will catch Goldilocks if they arrive home no more than 15 minutes after she wakes up. Find the probability that Goldilocks will escape without being caught.

p14. While sleeping in Mama Bear's bed, Goldilocks dreams about different colored sheep. She sees two blue sheep, two red sheep, and one green sheep. How many ways are there to arrange the sheep in a line if no two sheep of the same color can be standing next to each other? Note that sheep of the same color are indistinguishable.

p15. A bored and hungry Goldilocks finds an infinite number of raisins in her pocket. On the kitchen table lies an infinite number of bowls of porridge. She labels the bowls, numbering the leftmost bowl 1, the second leftmost bowl 2, and so on. She then distributes her raisins, starting by adding 1raisin to Bowl 1. Next, if she adds r raisins to Bowl b, then she will add r raisins to Bowl $2b$, and $r+1$ raisins to Bowl $2b+1$. Into how many of the 22019 leftmost bowls of porridge will Goldilocks add exactly 2017 raisins?

PS. You should use hide for answers.

– Individual Round

Individual **p1.** Compute $2020 \cdot \left(2^{(0\cdot 1)} + 9 - \frac{(20^1)}{8}\right)$.

p2. Nathan has five distinct shirts, three distinct pairs of pants, and four distinct pairs of shoes. If an "outfit" has a shirt, pair of pants, and a pair of shoes, how many distinct outfits can Nathan make?

p3. Let $ABCD$ be a rhombus such that $\triangle\ ABD$ and $\triangle\ BCD$ are equilateral triangles. Find the angle measure of $\angle ACD$ in degrees.

p4. Find the units digit of 2019^{2019}.

p5. Determine the number of ways to color the four vertices of a square red, white, or blue if two colorings that can be turned into each other by rotations and reflections are considered the same.

p6. Kathy rolls two fair dice numbered from 1 to 6. At least one of them comes up as a 4 or 5. Compute the probability that the sumof the numbers of the two dice is at least 10.

p7. Find the number of ordered pairs of positive integers (x, y) such that $20x + 19y = 2019$.

p8. Let p be a prime number such that both $2p-1$ and $10p-1$ are prime numbers. Find the sum of all possible values of p.

p9. In a square $ABCD$ with side length 10, let E be the intersection of AC and BD. There is a circle inscribed in triangle ABE with radius r and a circle circumscribed around triangle ABE with radius R. Compute $R - r$.

p10. The fraction $\frac{13}{37 \cdot 77}$ can be written as a repeating decimal $0.a_1a_2...a_{n-1}a_n$ with n digits in its shortest repeating decimal representation. Find $a_1 + a_2 + ... + a_{n-1} + a_n$.

p11. Let point E be the midpoint of segment AB of length 12. Linda the ant is sitting at A. If there is a circle O of radius 3 centered at E, compute the length of the shortest path Linda can take from A to B if she can't cross the circumference of O.

p12. Euhan and Minjune are playing tennis. The first one to reach 25 points wins. Every point ends with Euhan calling the ball in or out. If the ball is called in, Minjune receives a point. If the ball is called out, Euhan receives a point. Euhan always makes the right call when the ball is out. However, he has a $\frac34$ chance of making the right call when the ball is in, meaning that he has a $\frac14$ chance of calling a ball out when it is in. The probability that the ball is in is equal to the probability that the ball is out. If Euhan won, determine the expected number of wrong callsmade by Euhan.

p13. Find the number of subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ which contain four consecutive numbers.

p14. Ezra and Richard are playing a game which consists of a series of rounds. In each round, one of either Ezra or Richard receives a point. When one of either Ezra or Richard has three more points than the other, he is declared the winner. Find the number of games which last eleven rounds. Two games are considered distinct if there exists a round in which the two games had different outcomes.

p15. There are 10 distinct subway lines in Boston, each of which consists of a path of stations. Using any 9 lines, any pair of stations are connected. However, among any 8 lines there exists a pair of stations that cannot be reached from one another. It happens that the number of stations is minimized so this property is satisfied. What is the average number of stations that each line passes through?

p16. There exist positive integers k and $3 \nmid m$ for which

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{53} - \frac{1}{54} + \frac{1}{55} = \frac{3^k \times m}{28 \times 29 \times ... \times 54 \times 55}.$$

Find the value k.

p17. Geronimo the giraffe is removing pellets from a box without replacement. There are 5 red pellets, 10 blue pellets, and 15 white pellets. Determine the probability that all of the red pellets are removed before all the blue pellets and before all of the white pellets are removed.

p18. Find the remainder when

$$70!\left(\frac{1}{4\times 67}+\frac{1}{5\times 66}+...+\frac{1}{66\times 5}+\frac{1}{67\times 4}\right)$$

is divided by 71.

p19. Let $A_1A_2...A_{12}$ be the regular dodecagon. Let X be the intersection of A_1A_2 and A_5A_{11}. Given that $XA_2 \cdot A_1A_2 = 10$, find the area of dodecagon.

p20. Evaluate the following infinite series:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\sec^2 m - m\tan^2 n}{3^{m+n}(m+n)}$$

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PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Guts Round

– Round 1

p1. Alice has a pizza with eight slices. On each slice, she either adds only salt, only pepper, or leaves it plain. Determine how many ways there are for Alice to season her entire pizza.

p2. Call a number almost prime if it has exactly three divisors. Find the number of almost prime numbers less than 100.

p3. Determine the maximum number of points of intersection between a circle and a regular pentagon.

Round 2

p4. Let $d(n)$ denote the number of positive integer divisors of n. Find $d(d(20^{18}))$.

p5. 20 chubbles are equal to 19 flubbles. 20 flubbles are equal to 18 bubbles. How many bubbles are 1000 chubbles worth?

p6. Square $ABCD$ and equilateral triangle EFG have equal area. Compute $\frac{AB}{EF}$.

Round 3

p7. Find the minimumvalue of y such that $y = x^2 - 6x - 9$ where x is a real number.

p8. I have 2 pairs of red socks, 5 pairs of white socks, and 7 pairs of blue socks. If I randomly pull out one sock at a time without replacement, how many socks do I need to draw to guarantee that I have drawn 3 pairs of socks of the same color?

p9. There are 23 paths from my house to the school, 29 paths from the school to the library, and 3 paths fromthe library to town center. Additionally, there are 6 paths directly from my house to the library. If I have to pass through the library to get to town center, howmany ways are there to travel from my house all the way to the town center?

Round 4

p10. A circle of radius 25 and a circle of radius 4 are externally tangent. A line is tangent to the circle
of radius 25 at A and the circle of radius 4 at B, where $A \ne B$. Compute the length of AB.

p11. A gambler spins two wheels, one numbered 1 to 4 and another numbered 1 to 5, and the amount of money he wins is the sum of the two numbers he spins in dollars. Determine the expected amount of money he will win.

p12. Find the remainder when 20^{19} is divided by 18.

PS. You should use hide for answers. Rounds 5-8 have been posted here (`https://artofproblemsolving.com/community/c3h3166012p28809547`) and 9-12 here (`https://artofproblemsolving.com/community/c3h3166099p28810427`).Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 5

p13. Two concentric circles have radii 1 and 3. Compute the length of the longest straight line segment that can be drawn froma point on the circle of radius 1 to a point on the circle of radius 3 if the segment cannot intersect the circle of radius 1.

p14. Find the value of $\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\frac{5}{243}+...$

p15. Bob is trying to type the word ”welp”. However, he has a 18 chance ofmistyping each letter and instead typing one of four adjacent keys, each with equal probability. Find the probability that he types ”qelp” instead of ”welp”.

Round 6

p16. How many ways are there to tile a 2×12 board using an unlimited supply of 1×1 and 1×3 pieces?

p17. Jeffrey and Yiming independently choose a number between 0 and 1 uniformly at random. What is the probability that their two numbers can formthe sidelengths of a triangle with longest side of length 1?

p18. On $\triangle ABC$ with $AB = 12$ and $AC = 16$, let M be the midpoint of BC and E,F be the points such that E is on AB, F is on AC, and $AE = 2AF$. Let G be the intersection of EF and AM. Compute $\frac{EG}{GF}$.

Round 7

p19. Find the remainder when $2019x^{2019} - 2018x^{2018} + 2017x^{2017} - ... + x$ is divided by $x + 1$.

p20. Parallelogram $ABCD$ has $AB = 5$, $BC = 3$, and $\angle ABC = 45^o$. A line through C intersects AB at M and AD at N such that $\triangle BCM$ is isosceles. Determine the maximum possible area of $\triangle MAN$.

p21. Determine the number of convex hexagons whose sides only lie along the grid shown below.
`https://cdn.artofproblemsolving.com/attachments/2/9/93cf897a321dfda282a14e8f1c78d32fafb58 png`

Round 8

p22. Let $\triangle ABC$ be a triangle with side lengths $AB = 4$, $BC = 5$, and $CA = 6$. Extend ray $\overrightarrow{AB}$ to a point D such that $AD = 12$, and similarly extend ray $\overrightarrow{CB}$ to point E such that $CE = 15$. Let M and N be points on the circumcircles of ABC and DBE, respectively, such that line MN is tangent to both circles. Determine the length of MN.

p23. A volcano will erupt with probability $\frac{1}{20-n}$ if it has not erupted in the last n years. Determine the expected number of years between consecutive eruptions.

p24. If x and y are integers such that $x + y = 9$ and $3x^2 + 4xy = 128$, find x.

PS. You should use hide for answers. Rounds 1-4 have been posted here (`https://artofproblemsolving.com/community/c3h3165997p28809441`) and 9-12 here (`https://artofproblemsolving.com/community/c3h3166099p28810427`).Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 9

p25. Circle ω_1 has radius 1 and diameter AB. Let circle ω_2 be a circle withm aximum radius such that it is tangent to AB and internally tangent to ω_1. A point C is then chosen such that ω_2 is the incircle of triangle ABC. Compute the area of ABC.

p26. Two particles lie at the origin of a Cartesian plane. Every second, the first particle moves from its initial position (x, y) to one of either $(x + 1, y + 2)$ or $(x - 1, y - 2)$, each with probability 0.5. Likewise, every second the second particle moves from it's initial position (x, y) to one of either $(x + 2, y + 3)$ or $(x - 2, y - 3)$, each with probability 0.5. Let d be the distance distance between the two particles after exactly one minute has elapsed. Find the expected value of d^2.

p27. Find the largest possible positive integer n such that for all positive integers k with $gcd(k, n) = 1$, $k^2 - 1$ is a multiple of n.

Round 10

p28. Let $\triangle ABC$ be a triangle with side lengths $AB = 13$, $BC = 14$, $CA = 15$. Let H be the orthcenter of $\triangle ABC$, M be the midpoint of segment BC, and F be the foot of altitude from C to AB. Let K be the point on line BC such that $\angle MHK = 90^o$. Let P be the intersection of HK and AB. Let Q be the intersection of circumcircle of $\triangle FPK$ and BC. Find the length of QK.

p29. Real numbers (x, y, z) are chosen uniformly at random from the interval $[0, 2\pi]$. Find the probability that

$$\cos(x)\cdot\cos(y)+\cos(y)\cdot\cos(z)+\cos(z)\cdot\cos(x)+\sin(x)\cdot\sin(y)+\sin(y)\cdot\sin(z)+\sin(z)\cdot\sin(x)+1$$

is positive.

p30. Find the number of positive integers where each digit is either 1, 3, or 4, and the sum of the digits is 22.

Round 11

p31. In $\triangle\ ABC$, let D be the point on ray $\overrightarrow{CB}$ such that $AB = BD$ and let E be the point on ray $\overrightarrow{AC}$ such that $BC = CE$. Let L be the intersection of AD and circumcircle of $\triangle\ ABC$. The exterior angle bisector of $\angle C$ intersects AD at K and it follows that $AK = AB + BC + CA$. Given that points B, E, and L are collinear, find $\angle CAB$.

p32. Let a be the largest root of the equation $x^3 - 3x^2 + 10$. Find the remainder when $\lfloor a^{2019}\rfloor$ is divided by 17.

p33. For all $x, y \in Q$, functions $f, g, h : Q \to Q$ satisfy $f(x + g(y)) = g(h(f(x))) + y$. If $f(6) = 2$, $g\left(\frac{1}{2}\right) = 2$, and $h\left(\frac{7}{2}\right) = 2$, find all possible values of $f(2019)$.

Round 12

p34. An n-polyomino is formed by joining n unit squares along their edges. A free polyomino is a polyomino considered up to congruence. That is, two free polyominos are the same if there is a combination of translations, rotations, and reflections that turns one into the other. Let $P(n)$ be the number of free n-polyominos. For example, $P(3) = 2$ and $P(4) = 5$. Estimate $P(20)+P(19)$. If your estimate is E and the actual value is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 10 \cdot \left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$

p35. Estimate

$$\sum_{k=1}^{2019} sin(k),$$

where k is measured in radians. If your estimate is E and the actual value is A, your score for this problem will be $\max\left(0, 15 - 10 \cdot |E - A|\right)$.

p36. For a positive integer n, let $r_{10}(n)$ be the number of 10-tuples of (not necessarily positive) integers $(a_1, a_2, ..., a_9, a_{10})$ such that

$$a_1^2 + a_2^2 + ... + a_9^2 + a_{10}^2 = n.$$

Estimate $r_{10}(20) + r_{10}(19)$. If your estimate is E and the actual value is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 10 \cdot \left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$

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