Lexington Math Tournament

www.artofproblemsolving.com/community/c2812359

by parmenides51

– Team Round

Team Round **p1.** What is the smallest possible value for the product of two real numbers that differ by ten?

p2. Determine the number of positive integers n with $1 \le n \le 400$ that satisfy the following: • n is a square number. • n is one more than a multiple of 5. • n is even.

p3. How many positive integers less than 2019 are either a perfect cube or a perfect square but not both?

p4. Felicia draws the heart-shaped figure $GOAT$ that is made of two semicircles of equal area and an equilateral triangle, as shown below. If $GO = 2$, what is the area of the figure?
`https://cdn.artofproblemsolving.com/attachments/3/c/388daa657351100f408ab3f1185f9ab32fcca png`

p5. For distinct digits A, B, and C:

	A	A
	B	B
+	C	C
A	B	C

Compute $A \cdot B \cdot C$.

p6 What is the difference between the largest and smallest value for $lcm(a, b, c)$, where a, b, and c are distinct positive integers between 1 and 10, inclusive?

p7. Let A and B be points on the circumference of a circle with center O such that $\angle AOB = 100^o$. If X is the midpoint of minor arc AB and Y is on the circumference of the circle such that $XY \perp AO$, find the measure of $\angle OBY$.

p8. When Ben works at twice his normal rate and Sammy works at his normal rate, they can finish a project together in 6 hours. When Ben works at his normal rate and Sammy works as three times his normal rate, they can finish the same project together in 4 hours. How many hours does it take Ben and Sammy to finish that project if they each work together at their normal rates?

[b]p9.[/b] How many positive integer divisors n of 20000 are there such that when 20000 is divided by n, the quotient is divisible by a square number greater than 1?

p10. What's the maximum number of Friday the 13th's that can occur in a year?

p11. Let circle ω pass through points B and C of triangle ABC. Suppose ω intersects segment AB at a point $D \ne B$ and intersects segment AC at a point $E \ne C$. If $AD = DC = 12$, $DB = 3$, and $EC = 8$, determine the length of EB.

p12. Let a, b be integers that satisfy the equation $2a^2 - b^2 + ab = 18$. Find the ordered pair (a, b).

p13. Let a, b, c be nonzero complex numbers such that $a - \frac{1}{b} = 8, b - \frac{1}{c} = 10, c - \frac{1}{a} = 12.$
Find $abc - \frac{1}{abc}$.

p14. Let $\triangle ABC$ be an equilateral triangle of side length 1. Let ω_0 be the incircle of $\triangle ABC$, and for $n > 0$, define the infinite progression of circles ω_n as follows: • ω_n is tangent to AB and AC and externally tangent to ω_{n-1}. • The area of ω_n is strictly less than the area of ω_{n-1}.
Determine the total area enclosed by all ω_i for $i \ge 0$.

p15. Determine the remainder when $13^{2020} + 11^{2020}$ is divided by 144.

p16. Let x be a solution to $x + \frac{1}{x} = 1$. Compute $x^{2019} + \frac{1}{x^{2019}}$.

p17. The positive integers are colored black and white such that if n is one color, then $2n$ is the other color. If all of the odd numbers are colored black, then how many numbers between 100 and 200 inclusive are colored white?

p18. What is the expected number of rolls it will take to get all six values of a six-sided die face-up at least once?

p19. Let $\triangle ABC$ have side lengths $AB = 19$, $BC = 2019$, and $AC = 2020$. Let D, E be the feet of the angle bisectors drawn from A and B, and let X, Y to be the feet of the altitudes from C to AD and C to BE, respectively. Determine the length of XY .

p20. Suppose I have 5 unit cubes of cheese that I want to divide evenly amongst 3 hungry mice. I can cut the cheese into smaller blocks, but cannot combine blocks into a bigger block. Over all possible choices of cuts in the cheese, what's the largest possible volume of the smallest block of cheese?

PS. You had better use hide for answers.

– Theme Round

1 Joe Quigley

– **This section was written in memory of Joseph "Joe" John IV Quigley, who passed away last October. Joe Quigley ran the Math Club in Lexington for over twenty-four years, making math accessible and fun for students of all ages and abilities. His love of math was only eclipsed by his love for teaching, and he will be greatly missed by the entire Lexington community for his humor, patience, and dedication to his students.**

p1. Joe Quigley writes the following expression on the board for his students to evaluate:

$$2 \times 4 + 3 - 1$$

However, his students have not learned their order of operations so they randomly choose which operations to perform first, second, and third. How many different results can the students obtain?

p2. Joe Quigley flies airplanes around the Cartesian plane. There are two fuel stations, one at $(8, 12)$ and another at $(25, 5)$. He must station his home on the x-axis, and wants to be the same distance away from both station. Compute this distance.

p3. Joe Quigley has 12 students in his math class. He will distribute N worksheets among the students. Find the smallest positive integer N for which any such distribution of the N worksheets among the 12 students results in at least one student having at least 3 worksheets.

p4. Joe Quigley writes the number $4^{4^{4^4}}$ on the board. He correctly observes that this number has $2^a + b$ factors, where a and b are positive integers. Find the maximum possible value of a.

p5. Joe Quigley is teaching his students geometric series and asks them to compute the value of the following series:

$$\sum_{x \ge 1} \frac{x(x+1)}{2^x}$$

Compute this value.

PS. You should use hide for answers.

2 Astronomy

– **p6.** Alex and Anka go to see stars. Alex counts thirteen fewer stars than Anka, and both of the numbers of stars they counted are square numbers. It is known that exactly eight stars were counted by both of them. How many distinct stars did they count in total?

p7. Three planets with coplanar, circular, and concentric orbits are shown on the backside of this page. The radii of the three circles are $3, 4$, and 5. Initially, the three planets are collinear. Every hour, the outermost planet moves one-sixth of its full orbit, the middle planet moves one-fourth of its full orbit, and the innermost planet moves one-third of its full orbit (A full orbit occurs when a planet returns to its initial position). Moreover, all three planets orbit in the same direction. After three hours, what is the area of the triangle formed by the planets as its three vertices?
`https://cdn.artofproblemsolving.com/attachments/3/2/b1f7cc5ee1108d5ba68d0464df18782243630 png`

p8. Planets X and Y are following circular orbits around the same star. It takes planet X 120hours to complete a full trip around the star, and it takes planet Y 18 hours to complete a full trip around the star. If both planets begin at the same location in their orbit and travel clockwise, how many times will planet Y pass planet X in the time it takes planet X to complete a full trip around the star?

p9. In a certain stellar system, four asteroids form a rectangle. Your spaceship lies in the rectangle formed. The distances between three of these asteroids and your space ship are 1 light minute, 8 light minutes, and 4 light minutes. If the distance between your space ship and the last asteroid is an integer number of light minutes away then how far away from the last asteroid is your space ship?

p10. Two lost lovers, Laxe and Kaan, are both standing on the equators of planets with radius 13 miles. The center of the planets are 170 miles apart. At some time, both of them are as close to each other as possible. The planets rotate in opposite directions of each other at the same rate. What is the maximum possible distance between Laxe and Kaan such that they are still able to see each other?

PS. You should use hide for answers.

3 Holidays

– **p11.** Festivus occurs every year on December 23rd. In 2019, Festivus will occur on a Monday. On what day will Festivus occur in the year 2029?

p12. Leakey, Marpeh, Arik, and Yehau host a Secret Santa, where each one of them is assigned to give a present to somebody other than themselves. How many ways can the gifting be assigned such that everyone receives exactly one gift?

p13. How many permutations of the word C HR I ST M AS are there such that the S's are not next to each other and there is not a vowel anywhere between the two S's?

p14. Dasher, Dancer, Prancer, Vixen, Comet, Cupid, Donner, Blitzen, and Rudolph (9 reindeer) are guiding Santa's sleigh. They are arranged in a 3×3 array. You, the elf, have a big responsibility. You must place Santa's reindeer in a manner so that all of Santa's requests are met: • Donner is forgetful and must be put in the back row so Santa can keep an eye on Donner. • Additionally, Rudolph's big red nose distracts Donner, so Rudolph and Donner cannot be in the same column. • Finally, Comet is the fastest and must be put in the front row.
How many options do you have for arranging Santa's reindeer?

p15. Marpeh has a Christmas tree in the perfect shape of a right circular cone. The tree has base radius 8 inches and slant height 32 inches. He wants to place 3 ornaments on the surface of the tree with the following rules: • The red ornament is placed at the top of the tree. • The yellow ornament is placed along the circumference of the base of the tree. • The blue ornament is placed such that it is the same distance from the red and yellow ornaments when traveling on the surface of the tree.
What is the furthest possible surface distance that the blue ornament could be from the red ornament?

PS. You should use hide for answers.

– Individual Round

Individual **p1.** For positive real numbers x, y, the operation $\otimes$ is given by $x \otimes y = \sqrt{x^2 - y}$ and the operation $\oplus$ is given by $x \oplus y = \sqrt{x^2 + y}$. Compute $(((5 \otimes 4) \oplus 3) \otimes 2) \oplus 1$.

p2. Janabel is cutting up a pizza for a party. She knows there will either be 4, 5, or 6 people at the party including herself, but can't remember which. What is the least number of slices Janabel can cut her pizza to guarantee that everyone at the party will be able to eat an equal number of slices?

p3. If the numerator of a certain fraction is added to the numerator and the denominator, the result is $\frac{20}{19}$. What is the fraction?

p4. Let trapezoid $ABCD$ be such that $AB \parallel CD$. Additionally, $AC = AD = 5$, $CD = 6$, and $AB = 3$. Find BC.

p5. AtMerrick's Ice Cream Parlor, customers can order one of three flavors of ice cream and can have their ice cream in either a cup or a cone. Additionally, customers can choose any combination of the following three toppings: sprinkles, fudge, and cherries. How many ways are there to buy ice cream?

p6. Find the minimum possible value of the expression $|x + 1| + |x - 4| + |x - 6|$.

p7. How many 3 digit numbers have an even number of even digits?

p8. Given that the number $1a99b67$ is divisible by 7, 9, and 11, what are a and b? Express your answer as an ordered pair.

p9. Let O be the center of a quarter circle with radius 1 and arc AB be the quarter of the circle's circumference. Let M,N be the midpoints of AO and BO, respectively. Let X be the intersection of AN and BM. Find the area of the region enclosed by arc AB, AX,BX.

p10. Each square of a 5-by-1 grid of squares is labeled with a digit between 0 and 9, inclusive, such that the sum of the numbers on any two adjacent squares is divisible by 3. How many such labelings are possible if each digit can be used more than once?

p11. A two-digit number has the property that the difference between the number and the sum of its digits is divisible by the units digit. If the tens digit is 5, how many different possible values of the units digit are there?

p12. There are 2019 red balls and 2019 white balls in a jar. One ball is drawn and replaced with a ball of the other color. The jar is then shaken and one ball is chosen. What is the probability that this ball is red?

p13. Let $ABCD$ be a square with side length 2. Let ℓ denote the line perpendicular to diagonal

AC through point C, and let E and F be themidpoints of segments BC and CD, respectively. Let lines AE and AF meet ℓ at points X and Y , respectively. Compute the area of $\triangle AXY$.

p14. Express $\sqrt{21-6\sqrt6}+\sqrt{21+6\sqrt6}$ in simplest radical form.

p15. Let $\triangle ABC$ be an equilateral triangle with side length two. Let D and E be on AB and AC respectively such that $\angle ABE = \angle ACD = 15^o$. Find the length of DE.

p16. 2018 ants walk on a line that is 1 inch long. At integer time t seconds, the ant with label $1 \le t \le 2018$ enters on the left side of the line and walks among the line at a speed of $\frac{1}{t}$ inches per second, until it reaches the right end and walks off. Determine the number of ants on the line when $t = 2019$ seconds.

p17. Determine the number of ordered tuples $(a_1, a_2, ..., a_5)$ of positive integers that satisfy $a_1 \le a_2 \le ... \le a_5 \le 5$.

p18. Find the sum of all positive integer values of k for which the equation

$$\gcd(n^2-n-2019, n+1) = k$$

has a positive integer solution for n.

p19. Let $a_0 = 2$, $b_0 = 1$, and for $n \ge 0$, let

$$a_{n+1} = 2a_n + b_n + 1,$$

$$b_{n+1} = a_n + 2b_n + 1.$$

Find the remainder when a_{2019} is divided by 100.

p20. In $\triangle ABC$, let AD be the angle bisector of $\angle BAC$ such that D is on segment BC. Let T be the intersection of ray $\overrightarrow{CB}$ and the line tangent to the circumcircle of $\triangle ABC$ at A. Given that $BD = 2$ and $TC = 10$, find the length of AT.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Guts Round

– Round 1

p1. A positive integer is said to be transcendent if it leaves a remainder of 1 when divided by 2. Find the 1010th smallest positive integer that is transcendent.

p2. The two diagonals of a square are drawn, forming four triangles. Determine, in degrees, the sum of the interior angle measures in all four triangles.

p3. Janabel multiplied 2 two-digit numbers together and the result was a four digit number. If the thousands digit was nine and hundreds digit was seven, what was the tens digit?

Round 2

p4. Two friends, Arthur and Brandon, are comparing their ages. Arthur notes that 10 years ago, his age was a third of Brandon's current age. Brandon points out that in 12 years, his age will be double of Arthur's current age. How old is Arthur now?

p5. A farmer makes the observation that gathering his chickens into groups of 2 leaves 1 chicken left over, groups of 3 leaves 2 chickens left over, and groups of 5 leaves 4 chickens left over. Find the smallest possible number of chickens that the farmer could have.

p6. Charles has a bookshelf with 3 layers and 10 indistinguishable books to arrange. If each layer must hold less books than the layer below it and a layer cannot be empty, how many ways are there for Charles to arrange his 10 books?

Round 3

p7. Determine the number of factors of 2^{2019}.

p8. The points A, B, C, and D lie along a line in that order. It is given that $\overline{AB} : \overline{CD} = 1 : 7$ and $\overline{AC} : \overline{BD} = 2 : 5$. If $BC = 3$, find AD.

p9. A positive integer n is equal to one-third the sum of the first n positive integers. Find n.

Round 4

p10. Let the numbers a, b, c, and d be in arithmetic progression. If $a + 2b + 3c + 4d = 5$ and $a = \frac{1}{2}$

, find $a+b+c+d$.

p11. Ten people playing brawl stars are split into five duos of 2. Determine the probability that Jeff and Ephramare paired up.

p12. Define a sequence recursively by $F_0 = 0$, $F_1 = 1$, and for all $n \ge 2$,

$$F_n = \left\lceil \frac{F_{n-1} + F_{n-2}}{2} \right\rceil + 1,$$

where $\lceil r \rceil$ denotes the least integer greater than or equal to r . Find F_{2019}.

PS. You should use hide for answers. Rounds 5-8 have been posted here (https://artofproblemsolving.com/community/c3h3166019p28809679) and 9-12 here (https://artofproblemsolving.com/community/c3h3166115p28810631).Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Round 5

p13. Determine the number of different circular bracelets can be made with 7 beads, all either colored red or black.

p14. The product of 260 and n is a perfect square. The 2020th least possible positive integer value of n can be written as$p_1^{e1} \cdot p_2^{e2} \cdot p_3^{e3} \cdot p_4^{e4}$. Find the sum $p_1 + p_2 + p_3 + p_4 + e_1 + e_2 +_e 3 + e_4$.

p15. Let B and C be points along the circumference of circle ω. Let A be the intersection of the tangents at B and C and let $D \ne A$ be on $\overrightarrow{AC}$ such that $AC = CD = 6$. Given $\angle BAC = 60^o$, find the distance from point D to the center of ω.

Round 6

p16. Evaluate $\sqrt{2 + \sqrt{2 + \sqrt{2 + ...}}}$.

p17. Let $n(A)$ be the number of elements of set A and $||A||$ be the number of subsets of set A. Given that $||A|| + 2||B|| = 2^{2020}$, find the value of $n(B)$.

p18. a and b are positive integers and 8^a9^b has 578 factors. Find ab.

Round 7

p19. Determine the probability that a randomly chosen positive integer is relatively prime to 2019.

p20. A 3-by-3 grid of squares is to be numbered with the digits 1 through 9 such that each number is used once and no two even-numbered squares are adjacent. Determine the number of ways to number the grid.

p21. In $\triangle ABC$, point D is on AC so that $\frac{AD}{DC} = \frac{1}{13}$. Let point E be on BC, and let F be the intersection of AE and BD. If $\frac{DF}{FB} = \frac{2}{7}$ and the area of $\triangle DBC$ is 26, compute the area of $\triangle FAB$.

Round 8

p22. A quarter circle with radius 1 is located on a line with its horizontal base on the line and to the left of the vertical side. It is then rolled to the right until it reaches its original orientation. Determine the distance traveled by the center of the quarter circle.

p23. In 1734, mathematician Leonhard Euler proved that $\frac{\pi^2}{6} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...$. With this in mind, calculate the value of $\frac{1}{1} - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + ...$ (the series obtained by negating every other term of the original series).

p24. Billy the biker is competing in a bike show where he can do a variety of tricks. He knows that one trick is worth 2 points, 1 trick is worth 3 points, and 1 is worth 5 points, but he doesn't remember which trick is worth what amount. When it's Billy's turn to perform, he does 6 tricks, randomly choosing which trick to do. Compute the sum of all the possible values of points that Billy could receive in total.

PS. You should use hide for answers. Rounds 1-4 have been posted here (`https://artofproblemsolving.com/community/c3h3166016p28809598`) and 9-12 here (`https://artofproblemsolving.com/community/c3h3166115p28810631`).Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 9

p25. Find the largest prime factor of 1031301.

p26. Let $ABCD$ be a trapezoid such that $AB \parallel CD$, $\angle ABC = 90^o$, $AB = 5$, $BC = 20$, $CD = 15$. Let X, Y be the intersection of the circle with diameter BC and segment AD. Find the length of XY.

p27. A string consisting of 1's, 2's, and 3's is said to be a superpermutation of the string 123 if it contains every permutation of 123 as a contiguous substring. Find the smallest possible length of such a superpermutation.

Round 10

p28. Suppose that we have a function $f(x) = x^3 - 3x^2 + 3x$, and for all $n \ge 1$, $f^n(x)$ is defined by the function f applied n times to x. Find the remainder when $f^5(2019)$ is divided by 100.

p29. A function $f : 1, 2, ..., 10 \to 1, 2, ..., 10$ is said to be happy if it is a bijection and for all $n \in 1, 2, ..., 10$, $|n - f(n)| \le 1$. Compute the number of happy functions.

p30. Let $\vartriangle LMN$ have side lengths $LM = 15$, $MN = 14$, and $NL = 13$. Let the angle bisector of $\angle MLN$ meet the circumcircle of $\vartriangle LMN$ at a point $T \ne L$. Determine the area of $\vartriangle LMT$.

Round 11

p31. Find the value of
$$\sum_{d|2200} \tau(d),$$
where $\tau(n)$ denotes the number of divisors of n, and where $a|b$ means that $\frac{b}{a}$ is a positive integer.

p32. Let complex numbers $\omega_1, \omega_2, ..., \omega_{2019}$ be the solutions to the equation $x^{2019} - 1 = 0$. Evaluate
$$\sum_{i=1}^{2019} \frac{1}{1 + \omega_i}.$$

p33. Let M be a nonnegative real number such that $x^{x^{x^{\cdots}}}$ diverges for all $x > M$, and $x^{x^{x^{\cdots}}}$ converges for all $0 < x \le M$. Find M.

Round 12

p34. Estimate the number of digits in $\binom{2019}{1009}$. If your estimate is E and the actual value is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 10 \cdot \left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$

p35. You may submit any integer E from 1 to 30. Out of the teams that submit this problem, your score will be

$$\frac{E}{2\,(the\ number\ of\ teams\ who\ chose\ E)}$$

p36. We call a $m \times n$ domino-tiling a configuration of 2×1 dominoes on an $m \times n$ cell grid such that each domino occupies exactly 2 cells of the grid and all cells of the grid are covered. How many 8×8 domino-tilings are there? If your estimate is E and the actual value is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 10 \cdot \left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$

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