I Iberoamerican Interuniversitary Mathematics Competition - Colombia

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by Ozc

Problem 1 Prove that for any positive integer n the number $\left(\frac{3+\sqrt{17}}{2}\right)^n + \left(\frac{3-\sqrt{17}}{2}\right)^n$ is an odd integer.

Problem 2 Determine if for all natural n there is a $n \times n$ matrix of real entries such that its determinant is 0 and that changing any entry produce another matrix with nonzero determinant.

Problem 3 Let $r > n$ be positive integers. A "good word" is an n-tuple $\langle a_1, \ldots, a_n \rangle$ of distinct positive integers between 1 and r. A "play" consist of changing a integer a_i of a good word, in such a way that the resulting word is still a good word. The distance between two good words $A = \langle a_1, \ldots, a_n \rangle$ and $B = \langle b_1, \ldots, b_n \rangle$ is the minimun number of plays needed to obtain B from A. Find the maximun posible distance between two good words.

Problem 4 Let m be a line in the plane and M a point not in m. Find the locus of the focus of the parabolas with vertex M that are tangent to m.

Problem 5 Let $f : \mathbb{R} \to \mathbb{R}$, such that

i) For all $a \in \mathbb{R}$ and all $\epsilon > 0$, exists $\delta > 0$ such that $|x - a| < \delta \Rightarrow f(x) < f(a) + \epsilon$.

ii) For all $b \in \mathbb{R}$ and all $\epsilon > 0$, exists $x, y \in \mathbb{R}$ with $b - \epsilon < x < b < y < b + \epsilon$, such that $|f(x) - f(b)| < \epsilon$ and $|f(y) - f(b)| < \epsilon$.

Prove that if $f(a) < d < f(d)$ there exists c with $a < c < b$ or $b < c < a$ such that $f(c) = d$.

Problem 6 Let ϵ be an n-th root of the unity and suppose $z = p(\epsilon)$ is a real number where p is some polinomial with integer coefficients. Prove there exists a polinomial q with integer coefficients such that $z = q(2\cos(2\pi/n))$.
