II Iberoamerican Interuniversitary Mathematics Competition - Brazil

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by Ozc

Problem 1 Given two vectors $v = (v_1, \dots, v_n)$ and $w = (w_1 \dots, w_n)$ in $\mathbb{R}^n$, lets define $v * w$ as the matrix in which the element of row i and column j is $v_i w_j$. Supose that v and w are linearly independent. Find the rank of the matrix $v * w - w * v$.

Problem 2 In one side of a hall there are $2N$ rooms numbered from 1 to $2N$. In each room i between 1 and N there are p_i beds. Is needed to move every one of this beds to the roms from $N+1$ to $2N$, in such a way that for every j between $N+1$ and $2N$ the room j will have p_j beds. Supose that each bed can be move once and the price of moving a bed from room i to room j is $(i-j)^2$. Find a way to move every bed such that the total cost is minimize.

Note: The numbers p_i are given and satisfy that $p_1 + p_2 + \cdots + p_N = p_{N+1} + p_{N+2} + \cdots + p_{2N}$.

Problem 3 A set $X \subset \mathbb{R}$ has dimension zero if, for any $\epsilon > 0$ there exists a positive integer k and intervals $I_1, I_2, ..., I_k$ such that $X \subset I_1 \cup I_2 \cup \cdots \cup I_k$ with $\sum_{j=1}^{k} |I_j|^\epsilon < \epsilon$.

Prove that there exist sets $X, Y \subset [0, 1]$ both of dimension zero, such that $X + Y = [0, 2]$.

Problem 4 Let $f : [0, 1] \to [0, 1]$ a increasing continuous function, diferentiable in $(0, 1)$ and with derivative smaller than 1 in every point. The sequence of sets $A_1, A_2, A_3, \dots$ is define as: $A_1 = f([0, 1])$, and for $n \geq 2$, $A_n = f(A_{n-1})$. Prove that $\lim\limits_{n \to +\infty} d(A_n) = 0$, where $d(A)$ is the diameter of the set A.

Note: The diameter of a set X is define as $d(X) = \sup_{x,y \in X} |x - y|$.

Problem 5 Let n, d be integers with $n, k > 1$ such that $g.c.d(n, d!) = 1$. Prove that n and $n + d$ are primes if and only if

$$d!d((n-1)! + 1) + n(d! - 1) \equiv 0 \pmod{n(n + d)}.$$

Problem 6 A group is call locally cyclic if any finitely generated subgroup is cyclic. Prove that a locally cyclic group is isomorphic to one of its proper subgroups if and only if it's isomorphic to a proper subgroup of the rational numbers with the adition.
