Berkeley Math Tournament - Team Round, years 2012-19

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2012.1 Let S be the set of all rational numbers $x \in [0, 1]$ with repeating base 6 expansion

$$x = 0.\overline{a_1a_2...a_k} = 0.a_1a_2...a_ka_1a_2...a_k...$$

for some finite sequence $\{a_i\}_{i=1}^k$ of distinct nonnegative integers less than 6. What is the sum of all numbers that can be written in this form? (Put your answer in base 10.)

2012.2 Evaluate $\prod_{k=1}^{254} \log_{k+1}(k+2)^{u_k}$, where $u_k = \begin{cases} -k & \text{if } k \text{ is odd} \\ \frac{1}{k-1} & \text{if } k \text{ is even} \end{cases}$

2012.3 Let ABC be a triangle with side lengths $AB = 2011$, $BC = 2012$, $AC = 2013$. Create squares $S_1 = ABB'A''$, $S_2 = ACC''A'$, and $S_3 = CBB''C'$ using the sides AB, AC, BC respectively, so that the side $B'A''$ is on the opposite side of AB from C, and so forth. Let square S_4 have side length $A''A'$, square S_5 have side length $C''C'$, and square S_6 have side length $B''B'$. Let $A(S_i)$ be the area of square S_i . Compute $\frac{A(S_4)+A(S_5)+A(S_6)}{A(S_1)+A(S_2)+A(S_3)}$?

2012.4 There are 12 people labeled $1, ..., 12$ working together on 12 missions, with people $1, ..., i$working on the ith mission. There is exactly one spy among them. If the spy is not working on a mission, it will be a huge success, but if the spy is working on the mission, it will fail with probability $1/2$. Given that the first 11 missions succeed, and the 12th mission fails, what is the probability that person 12 is the spy?

2012.5 Let $p > 1$ be relatively prime to 10. Let n be any positive number and$d$ be the last digit of n. Define $f(n) = \lfloor \frac{n}{10} \rfloor + d \cdot m$. Then, we can call m a *divisibility multiplier* for p, if $f(n)$ is divisible by p if and only if n is divisible by p. Find a divisibility multiplier for 2013.

2012.6 A circle with diameter AB is drawn, and the point P is chosen on segment AB so that $\frac{AP}{AB} = \frac{1}{42}$. Two new circles a and b are drawn with diameters AP and PB respectively. The perpendicular line to AB passing through P intersects the circle twice at points S and T . Two more circles s and t are drawn with diameters SP and ST respectively. For any circle ω let $A(\omega)$ denote the area of the circle. What is $\frac{A(s)+A(t)}{A(a)+A(b)}$?

2012.7 Suppose Bob begins walking at a constant speed from point N to point S along the path indicated by the following figure.
https://cdn.artofproblemsolving.com/attachments/6/2/f5819267020f2bd38e52c6e873a2cf91ce8c4
png

After Bob has walked a distance of x, Alice begins walking at point N, heading towards point S along the same path. Alice walks 1.28 times as fast as Bob when they are on the same line segment and 1.06 times as fast as Bob otherwise. For what value of x do Alice and Bob meet at point S?

2012.8 Let ϕ be the Euler totient function. Let $\phi^k(n) = (\underbrace{\phi \circ ... \circ \phi}_{k})(n)$ be ϕ composed with itself k times.

Define $\theta(n) = min\{k \in N|\phi^k(n) = 1\}$
. For example, $\phi^1(13) = \phi(13) = 12$ $\phi^2(13) = \phi(\phi(13)) = 4$ $\phi^3(13) = \phi(\phi(\phi(13))) = 2$ $\phi^4(13) = \phi(\phi(\phi(\phi(13)))) = 1$
so $\theta(13) = 4$. Let $f(r) = \theta(13^r)$. Determine $f(2012)$.

2012.9 A permutation of a set is a bijection from the set to itself. For example, if σ is the permutation $17 \mapsto 3$, $2 \mapsto 1$, and $3 \mapsto 2$, and we apply it to the ordered triplet $(1, 2, 3)$, we get the reordered triplet $(3, 1, 2)$. Let σ be a permutation of the set $\{1, ..., n\}$. Let

$$\theta_k(m) = \begin{cases} m+1 & \text{for } m < k \\ 1 & \text{for } m = k \\ m & \text{for } m > k \end{cases}$$

Call a finite sequence $\{a_i\}_{i=1}^{j}$ a disentanglement of σ if $\theta_{a_j} \circ ... \circ \theta_{a_j} \circ \sigma$ is the identity permutation. For example, when $\sigma = (3, 2, 1)$, then $\{2, 3\}$ is a disentaglement of σ. Let $f(\sigma)$ denote the minimum number k such that there is a disentanglement of σ of length k. Let $g(n)$ be the expected value for $f(\sigma)$ if σ is a random permutation of $\{1, ..., n\}$. What is $g(6)$?

2012.10 Suppose that 728 coins are set on a table, all facing heads up at first. For each iteration, we randomly choose 314 coins and flip them (from heads to tails or vice versa). Let a/b be the expected number of heads after we finish 4001 iterations, where a and b are relatively prime. Find $a + b$ mod 10000.

2013.1 A time is called *reflexive* if its representation on an analog clock would still be permissible if the hour and minute hand were switched. In a given non-leap day ($12 : 00 : 00.00$ a.m. to $11 : 59 : 59.99$ p.m.), how many times are reflexive?

2013.2 Find the sum of all positive integers N such that $s = \sqrt[3]{2 + \sqrt{N}} + \sqrt[3]{2 - \sqrt{N}}$ is also a positive integer

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2013.3 A round robin tennis tournament is played among 4 friends in which each player plays every other player only one time, resulting in either a win or a loss for each player. If overall placement is determined strictly by how many games each player won, how many possible placements are there at the end of the tournament? For example, Andy and Bob tying for first and Charlie and Derek tying for third would be one possible case.

2013.4 Find the sum of all real numbers x such that $x^2 = 5x + 6\sqrt{x} - 3$.

2013.5 Circle C_1 has center O and radius OA, and circle C_2 has diameter OA. AB is a chord of circle C_1 and BD may be constructed with D on OA such that BD and OA are perpendicular. Let C be the point where C_2 and BD intersect. If $AC = 1$, find AB.

2013.6 In a class of 30 students, each students knows exactly six other students. (Of course, knowing is a mutual relation, so if A knows B, then B knows A). A group of three students is balanced if either all three students know each other, or no one knows anyone else within that group. How many balanced groups exist?

2013.7 Consider the infinite polynomial $G(x) = F_1x + F_2x^2 + F_3x^3 + ...$ defined for $0 < x < \frac{\sqrt{5}-1}{2}$ where Fk is the kth term of the Fibonacci sequence defined to be $F_k = F_{k-1} + F_{k-2}$ with $F_1 = 1$, $F_2 = 1$. Determine the value a such that $G(a) = 2$.

2013.8 A parabola has focus F and vertex V , where $VF =$ 10. Let AB be a chord of length 100 that passes through F. Determine the area of $\triangle VAB$.

2013.9 Sequences x_n and y_n satisfy the simultaneous relationships $x_k = x_{k+1} + y_{k+1}$ and $x_k > y_k$ for all $k \ge 1$. Furthermore, either $y_k = y_{k+1}$ or $y_k = x_{k+1}$. If $x_1 = 3 + \sqrt{2}$, $x_3 = 5 - \sqrt{2}$, and $y_1 = y_5$, evaluate

$$(y_1)^2 + (y_2)^2 + (y_3)^2 + ...$$

2013.10 In a far away kingdom, there exist k^2 cities subdivided into k distinct districts, such that in the i^{th} district, there exist $2i - 1$ cities. Each city is connected to every city in its district but no cities outside of its district. In order to improve transportation, the king wants to add $k - 1$ roads such that all cities will become connected, but his advisors tell him there are many ways to do this. Two plans are different if one road is in one plan that is not in the other. Find the total number of possible plans in terms of k.

2014.1 What is the value of $1 + 7 + 21 + 35 + 35 + 21 + 7 + 1$?

2014.2 A mathematician is walking through a library with twenty-six shelves, one for each letter of the alphabet. As he walks, the mathematician will take at most one book off each shelf. He likes symmetry, so if the letter of a shelf has at least one line of symmetry (e.g., M works, L does not), he will pick a book with probability $\frac{1}{2}$. Otherwise he has a $\frac{1}{4}$ probability of taking a book. What is the expected number of books that the mathematician will take?

2014.3 Together, Abe and Bob have less than or equal to \$ 100. When Corey asks them how much money they have, Abe says that the reciprocal of his money added to Bob's money is thirteen times as much as the sum of Abe's money and the reciprocal of Bob's money. If Abe and Bob both have integer amounts of money, how many possible values are there for Abe's money?

2014.4 In a right triangle, the altitude from a vertex to the hypotenuse splits the hypotenuse into two segments of lengths a and b. If the right triangle has area T and is inscribed in a circle of area C, find ab in terms of T and C.

2014.5 Call two regular polygons supplementary if the sum of an internal angle from each polygon adds up to 180^o. For instance, two squares are supplementary because the sum of the internal angles is $90^o + 90^o = 180^o$. Find the other pair of supplementary polygons. Write your answer in the form (m, n) where m and n are the number of sides of the polygons and $m < n$.

2014.6 A train is going up a hill with vertical velocity given as a function of t by $\frac{1}{1-t^4}$, where t is between $[0, 1)$. Determine its height as a function of t.

2014.7 Let $VWXYZ$ be a square pyramid with vertex V with height 1, and with the unit square as its base. Let $STANFURD$ be a cube, such that face $FURD$ lies in the same plane as and shares the same center as square face $WXYZ$. Furthermore, all sides of $FURD$ are parallel to the sides of $WXYZ$. Cube $STANFURD$ has side length s such that the volume that lies inside the cube but outside the square pyramid is equal to the volume that lies inside the square pyramid but outside the cube. What is the value of s?

2014.8 Annisa has n distinct textbooks, where $n > 6$. She has a different ways to pick a group of 4 books, b different ways to pick 5 books and c different ways to pick 6 books. If Annisa buys two more (distinct) textbooks, how many ways will she be able to pick a group of 6 books?

2014.9 Two different functions f, g of x are selected from the set of real-valued functions

$$\left\{sinx, e^{-x}, x\ln x, \arctan x, \sqrt{x^2+x} - \sqrt{x^2+x} - x, \frac{1}{x}\right\}$$

to create a product function $f(x)g(x)$. For how many such products is $\lim_{x\to infty} f(x)g(x)$ finite?

2014.10 A *unitary* divisor d of a number n is a divisor n that has the property $\gcd(d, n/d) = 1$. If $n = 1620$, what is the sum of all of the unitary divisors of d?

2014.11 Suppose that $x^{10} + x + 1 = 0$ and $x^100 = a_0 + a_1x + ... + a_9x^9$. Find a_5.

2014.12 A two-digit integer is *reversible* if, when written backwards in base 10, it has the same number of positive divisors. Find the number of reversible integers.

2014.13 Let ABC be a triangle with $AB = 16$, $AC = 10$, $BC = 18$. Let D be a point on AB such that $4AD = AB$ and let E be the foot of the angle bisector from B onto AC. Let P be the intersection of CD and BE. Find the area of the quadrilateral $ADPE$.

2014.14 Let (x, y) be an intersection of the equations $y = 4x^2-28x+41$ and $x^2+25y^2-7x+100y+\frac{349}{4} = 0$. Find the sum of all possible values of x.

2014.15 Suppose a box contains 28 balls: 1 red, 2 blue, 3 yellow, 4 orange, 5 purple, 6 green, and 7 pink. One by one, each ball is removed uniformly at random and without replacement until all 28 balls have been removed. Determine the probability that the most likely "scenario of exhaustion" occurs; that is, determine the probability that the first color to have all such balls removed from the box is red, that the second is blue, the third is yellow, the fourth is orange, the fifth is purple, the sixth is green, and the seventh is pink.

2014.16 Evaluate

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\min(n,k)\left(\frac{1}{2}\right)^n\left(\frac{1}{3}\right)^k$$

2014.17 Suppose you started at the origin on the number line in a coin-flipping game. Every time you flip a heads, you move forward one step, otherwise you move back one step. However, there are walls at positions 8 and -8; if you are at these positions and your coin flip dictates that you should move past them, instead you must stay. What is the expected number of coin flips needed to have visited both walls?

2014.18 Monty wants to play a game with you. He shows you five boxes, one of which contains a prize and four of which contain nothing. He allows you to choose one box but not to open it. He then opens one of the other four boxes that he knows to contain nothing. Then, he makes you switch and choose a different, unopened box. However, Monty sketchily reveals the contents of another (empty) box, selected uniformly at random from the two or three closed boxes (that you do not currently have chosen) that he knows to contain no prize. He then offers you the chance to switch again. Assuming you seek to maximize your return, determine the probability you get a prize.

2014.19 A number k is *nice* in base b if there exists a k-digit number n such that $n, 2n, ...kn$ are each some cyclic shifts of the digits of n in base b (for example, 2 is *nice* in base 5 because $2 \cdot 135 = 315$). Determine all nice numbers in base 18.

2014.20 A certain type of Bessel function has the form $I(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}d\theta$ for all real x. Evaluate $\int_0^{\infty} xI(2x)e^{-x^2}dx$.

2015.1 A fair 6-sided die is repeatedly rolled until a $1, 4, 5$, or 6 is rolled. What is the expected value of the product of all the rolls?

2015.2 Compute the sum of the digits of 1001^{10}

2015.3 How many ways are there to place the numbers $2, 3, ..., 10$ in a 3×3 grid, such that any two numbers that share an edge are mutually prime?

2015.4 Triangle ABC has side lengths $AB = 3$, $BC = 4$, and $CD = 5$. Draw line ℓ_A such that ℓ_A is parallel to BC and splits the triangle into two polygons of equal area. Define lines ℓ_B and ℓ_C analogously. The intersection points of ℓ_A, ℓ_B, and ℓ_C form a triangle. Determine its area.

2015.5 Determine the smallest positive integer containing only 0 and 1 as digits that is divisible by each integer 1 through 9.

2015.6 Consider the set $S = \{1, 2, ..., 2015\}$. How many ways are there to choose 2015 distinct (possibly empty and possibly full) subsets $X_1, X_2, ..., X_{2015}$ of S such that X_i is strictly contained in X_{i+1} for all $1 \le i \le 2014$?

2015.7 $X_1, X_2, ..., X_{2015}$ are 2015 points in the plane such that for all $1 \le i, j \le 2015$, the line segment $X_iX_{i+1} = X_jX_{j+1}$ and angle $\angle X_iX_{i+1}X_{i+2} = \angle X_jX_{j+1}X_{j+2}$ (with cyclic indices such that $X_{2016} = X_1$ and $X_{2017} = X_2$). Given fixed X_1 and X_2, determine the number of possible locations for X_3.

2015.8 The sequence $(x_n)_{n \in N}$ satisfies $x_1 = 2015$ and $x_{n+1} = \sqrt[3]{13x_n - 18}$ for all $n \ge 1$. Determine $\lim_{n \to \infty} x_n$.

2015.9 Find the side length of the largest square that can be inscribed in the unit cube.

2015.10 Quadratics $g(x) = ax^2 + bx + c$ and $h(x) = dx^2 + ex + f$ are such that the six roots of g, h, and $g - h$ are distinct real numbers (in particular, they are not double roots) forming an arithmetic progression in some order. Determine all possible values of a/d.

2015.11 Write down $1, 2, 3, ..., 2015$ in a row on a whiteboard. Every minute, select a pair of adjacent numbers at random, erase them, and insert their sum where you selected the numbers. (For instance, selecting 3 and 4 from $1, 2, 3, 4, 5$ would result in $1, 2, 7, 5$.) Repeat this process until you have two numbers remaining. What is the probability that the smaller number is less than or equal to 2015?

2015.12 Let $f(n)$ be the number of ordered pairs (k, ℓ) of positive integers such that $n = (2\ell - 1) \cdot 2^k - k$, and let $g(n)$ be the number of ordered pairs (k, ℓ) of positive integers such that $n = \ell \cdot 2^{k+1} - k$. Compute $\sum_{i=1}^{\infty} \frac{f(i) - g(i)}{2^i}$.

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2015.13 There exist right triangles with integer side lengths such that the legs differ by 1. For example, $3-4-5$ and $20-21-29$ are two such right triangles. What is the perimeter of the next smallest Pythagorean right triangle with legs differing by 1?

2015.14 Alice is at coordinate point $(0,0)$ and wants to go to point $(11,6)$. Similarly, Bob is at coordinate point $(5,6)$ and wants to go to point $(16,0)$. Both of them choose a lattice path from their current position to their target position at random (such that each lattice path has an equal probability of being chosen), where a lattice path is defined to be a path composed of unit segments with orthogonal direction (parallel to x-axis or y-axis) and of minimal length. (For instance, there are six lattice paths from $(0,0)$ to $(2,2)$.) If they walk with the same speed, find the probability that they meet.

2015.15 Compute

$$\int_{1/2}^{2} \frac{x^2+1}{x^2(x^{2015}+1)}dx.$$

2015.16 Five points A, B, C, D, and E in three-dimensional Euclidean space have the property that $AB = BC = CD = DE = EA = 1$ and $\angle ABC = \angle BCD = \angle CDE = \angle DEA = 90^o$. Find all possible $\cos(\angle EAB)$.

2015.17 There exist real numbers x and y such that $x(a^3+b^3+c^3)+3yabc \geq (x+y)(a^2b+b^2c+c^2a)$ holds for all positive real numbers a, b, and c. Determine the smallest possible value of x/y.
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2015.18 A value $x \in [0,1]$ is selected uniformly at random. A point (a,b) is called *friendly* to x if there exists a circle between the lines $y = 0$ and $y = 1$ that contains both (a,b) and $(0,x)$. Find the area of the region of the plane determined by possible locations of friendly points.

2015.19 Two sequences $(x_n)_{n\in N}$ and $(y_n)_{n\in N}$ are defined recursively as follows:

$x_0 = 2015$ and $x_{n+1} = \left\lfloor x_n \cdot \frac{y_{n+1}}{y_{n-1}} \right\rfloor$ for all $n \geq 0$,

$y_0 = 307$ and $y_{n+1} = y_n + 1$ for all $n \geq 0$.

Compute $\lim_{n\to\infty} \frac{x_n}{(y_n)^2}$.

2015.20 The Tower of Hanoi is a puzzle with n disks of different sizes and 3 vertical rods on it. All of the disks are initially placed on the leftmost rod, sorted by size such that the largest disk is on the bottom. On each turn, one may move the topmost disk of any nonempty rod onto any other rod, provided that it is smaller than the current topmost disk of that rod, if it exists. (For instance, if there were two disks on different rods, the smaller disk could move to either of the other two rods, but the larger disk could only move to the empty rod.) The puzzle is solved when all of the disks are moved to the rightmost rod. The specifications normally include an intelligent monk to move the disks, but instead there is a monkey making random moves (with each valid move having an equal probability of being selected). Given 64 disks, what is the expected number of moves the monkey will have to make to solve the puzzle?

2016.1 Define an such that $a_1 = \sqrt{3}$ and for all integers i, $a_{i+1} = a_i^2 - 2$. What is a_{2016}?

2016.2 Jennifer wants to do origami, and she has a square of side length 1. However, she would prefer to use a regular octagon for her origami, so she decides to cut the four corners of the square to get a regular octagon. Once she does so, what will be the side length of the octagon Jennifer obtains?

2016.3 A little boy takes a 12 in long strip of paper and makes a Mobius strip out of it by tapping the ends together after adding a half twist. He then takes a 1 inch long train model and runs it along the center of the strip at a speed of 12 inches per minute. How long does it take the train model to make two full complete loops around the Mobius strip? A complete loop is one that results in the train returning to its starting point.

2016.4 How many graphs are there on 6 vertices with degrees $1, 1, 2, 3, 4, 5$?

2016.5 Let ABC be a right triangle with $AB = BC = 2$. Let ACD be a right triangle with angle $\angle DAC = 30$ degrees and $\angle DCA = 60$ degrees. Given that ABC and ACD do not overlap, what is the area of triangle BCD?

2016.6 How many integers less than 400 have exactly 3 factors that are perfect squares?

2016.7 Suppose $f(x, y)$ is a function that takes in two integers and outputs a real number, such that it satisfies

$$f(x, y) = \frac{f(x, y+1) + f(x, y-1)}{2}$$
$$f(x, y) = \frac{f(x+1, y) + f(x-1, y)}{2}$$

What is the minimum number of pairs (x, y) we need to evaluate to be able to uniquely determine f?

2016.8 How many ways are there to divide 10 candies between 3 Berkeley students and 4 Stanford students, if each Berkeley student must get at least one candy? All students are distinguishable from each other; all candies are indistinguishable.

2016.9 How many subsets (including the empty-set) of $\{1, 2..., 6\}$ do not have three consecutive integers?

2016.10 What is the smallest possible perimeter of a triangle with integer coordinate vertices, area $\frac{1}{2}$, and no side parallel to an axis?

2016.11 Circles C_1 and C_2 intersect at points X and Y. Point A is a point on C_1 such that the tangent line with respect to C_1 passing through A intersects C_2 at B and C, with A closer to B than C, such that $2016 \cdot AB = BC$. Line XY intersects line AC at D. If circles C_1 and C_2 have radii of 20 and 16, respectively, find $\sqrt{1 + BC/BD}$.

2016.12 Consider a solid hemisphere of radius 1. Find the distance from its center of mass to the base.

2016.13 Consider an urn containing 51 white and 50 black balls. Every turn, we randomly pick a ball, record the color of the ball, and then we put the ball back into the urn. We stop picking when we have recorded n black balls, where n is an integer randomly chosen from $\{1, 2, ..., 100\}$. What is the expected number of turns?

2016.14 Consider the set of axis-aligned boxes in R^d , $B(a, b) = \{x \in R^d : \forall i, a_i \leq x_i \leq b_i\}$ where $a, b \in R^d$. In terms of d, what is the maximum number n, such that there exists a set of n points $S = \{x_1, ..., x_n\}$ such that no matter how one partition $S = P \cup Q$ with P, Q disjoint and P, Q can possibly be empty, there exists a box B such that all the points in P are contained in B, and all the points in Q are outside B?

2016.15 Let s_1, s_2, s_3 be the three roots of $x^3 + x^2 + \frac{9}{2}x + 9$.

$$\prod_{i=1}^{3}(4s_i^4 + 81)$$

can be written as $2^a 3^b 5^c$. Find $a + b + c$.

2017.1 You are racing an Artificially Intelligent Robot, called Al, that you built. You can run at a constant speed of 10 m/s throughout the race. Meanwhile, Al starts running at a constant speed of 1 m/s. Thereafter, when exactly 1 second has passed from when Al last changed its speed, Al's speed instantaneously becomes 1 m/s faster, so that Al runs at a constant speed of k m/s in the kth second of the race. (Start counting seconds at 1). Suppose Al beats you by exactly 1 second. How many meters was the race?

2017.2 Colin has 900 Choco Pies. He realizes that for some integer values of $n \leq 900$, if he eats n pies a day, he will be able to eat the same number of pies every day until he runs out. How many possible values of n are there?

2017.3 Suppose we have $w < x < y < z$, and each of the 6 pairwise sums are distinct. The 4 greatest sums are $4, 3, 2, 1$. What is the sum of all possible values of w?

2017.4 2 darts are thrown randomly at a circular board with center O, such that each dart has an equal probability of hitting any point on the board. The points at which they land are marked A and B. What is the probability that $\angle AOB$ is acute?

2017.5 You enter an elevator on floor 0 of a building with some other people, and request to go to floor 10. In order to be efficient, it doesn't stop at adjacent floors (so, if it's at floor 0, its next stop cannot be floor 1). Given that the elevator will stop at floor 10, no matter what other floors it stops at, how many combinations of stops are there for the elevator?

2017.6 The center of a square of side length 1 is placed uniformly at random inside a circle of radius 1. Given that we are allowed to rotate the square about its center, what is the probability that the entire square is contained within the circle for some orientation of the square?

2017.7 There are 86400 seconds in a day, which can be deduced from the conversions between seconds, minutes, hours, and days. However, the leading scientists decide that we should decide on 3 new integers x, y, and z, such that there are x seconds in a minute, y minutes in an hour, and z hours in a day, such that $xyz = 86400$ as before, but such that the sum $x + y + z$ is minimized. What is the smallest possible value of that sum?

2017.8 A function f with its domain on the positive integers $N = \{1, 2, ...\}$ satisfies the following conditions:
(a) $f(1) = 2017$.
(b) $\sum_{i=1}^{n} f(i) = n^2 f(n)$, for every positive integer $n > 1$.
What is the value of $f(2017)$?

2017.9 Let $AB = 10$ be a diameter of circle P. Pick point C on the circle such that $AC = 8$. Let the circle with center O be the incircle of $\triangle ABC$. Extend line AO to intersect circle P again at D. Find the length of BD.

2017.10 You and your friend play a game on a 7×7 grid of buckets. Your friend chooses 5 "lucky" buckets by marking an "X" on the bottom that you cannot see. However, he tells you that they either form a vertical, or horizontal line of length 5. To clarify, he will select either of the following sets of buckets:
either $\{(a, b), (a, b + 1), (a, b + 2), (a, b + 3), (a, b + 4)\}$,
or $\{(b, a), (b + 1, a), (b + 2, a), (b + 3, a), (b + 4, a)\}$,
with $1 \le a \le 7$, and $1 \le b \le 3$. Your friend lets you pick up at most n buckets, and you win if one of the buckets you picked was a "lucky" bucket. What is the minimum possible value of n such that, if you pick your buckets optimally, you can guarantee that at least one is "lucky"?

2017.11 Ben picks a positive number n less than 2017 uniformly at random. Then Rex, starting with the number 1, repeatedly multiplies his number by n and then finds the remainder when dividing by 2017. Rex does this until he gets back to the number 1. What is the probability that, during this process, Rex reaches every positive number less than 2017 before returning back to 1?

2017.12 A robot starts at the origin of the Cartesian plane. At each of 10 steps, he decides to move 1 unit in any of the following directions: left, right, up, or down, each with equal probability. After 10 steps, the probability that the robot is at the origin is $\frac{n}{4^{10}}$. Find$n$

2017.13 4 equilateral triangles of side length 1 are drawn on the interior of a unit square, each one of which shares a side with one of the 4 sides of the unit square. What is the common area enclosed by all 4 equilateral triangles?

2017.14 Suppose that there is a set of 2016 positive numbers, such that both their sum, and the sum of their reciprocals, are equal to 2017. Let x be one of those numbers. Find the maximum possible value of $x + \frac{1}{x}$.
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2017.15 In triangle ABC, the angle at C is 30^o, side BC has length 4, and side AC has length 5. Let P be the point such that triangle ABP is equilateral and non-overlapping with triangle ABC. Find the distance from C to P.

2018.1 A circle with radius 5 is inscribed in a right triangle with hypotenuse 34 as shown below. What is the area of the triangle? Note that the diagram is not to scale.

2018.2 For how many values of x does $20^x \cdot 18^x = 2018^x$?

2018.3 2018 people (call them $A, B, C, \ldots$) stand in a line with each permutation equally likely. Given that A stands before B, what is the probability that C stands after B?

2018.4 Consider a standard (8-by-8) chessboard. Bishops are only allowed to attack pieces that are along the same diagonal as them (but cannot attack along a row or column). If a piece can attack another piece, we say that the pieces threaten each other. How many bishops can you place a chessboard without any of them threatening each other?

2018.5 How many integers can be expressed in the form: $\pm 1 \pm 2 \pm 3 \pm 4 \pm \cdots \pm 2018$?

2018.6 A rectangular prism with dimensions 20 cm by 1 cm by 7 cm is made with blue 1 cm unit cubes. The outside of the rectangular prism is coated in gold paint. If a cube is chosen at random and rolled, what is the probability that the side facing up is painted gold?

2018.7 Suppose there are 2017 spies, each with $\frac{1}{2017}$th of a secret code. They communicate by telephone; when two of them talk, they share all information they know with each other. What is the minimum number of telephone calls that are needed for all 2017 people to know all parts of the code?

2018.8 Alice is playing a game with 2018 boxes, numbered $1 - 2018$, and a number of balls. At the beginning, boxes $1 - 2017$ have one ball each, and box 2018 has $2018n$ balls. Every turn, Alice chooses i and j with $i > j$, and moves exactly i balls from box i to box j. Alice wins if all balls end up in box 1. What is the minimum value of n so that Alice can win this game?

2018.9 Circles A, B, and C are externally tangent circles. Line PQ is drawn such that PQ is tangent to A at P, tangent to B at Q, and does not intersect with C. Circle D is drawn such that it passes through the centers of A, B, and C. Let R be the point on D furthest from PQ. If A, B, and C have radii 3, 2, and 1, respectively, the area of triangle PQR can be expressed in the form of $a + b\sqrt{c}$, where a, b, and c are integers with c not divisible by any prime square. What is $a + b + c$?

2018.10 A rectangular prism has three distinct faces of area 24, 30, and 32. The diagonals of each distinct face of the prism form sides of a triangle. What is the triangle's area?

2018.11 Ankit, Box, and Clark are playing a game. First, Clark comes up with a prime number less than 100. Then he writes each digit of the prime number on a piece of paper (writing 0 for the tens digit if he chose a single-digit prime), and gives one each to Ankit and Box, without telling them which digit is the tens digit, and which digit is the ones digit. The following exchange occurs:

1. Clark: There is only one prime number that can be made using those two digits.
2. Ankit: I don't know whether I'm the tens digit or the ones digit.
3. Box: I don't know whether I'm the tens digit or the ones digit.
4. Box: You don't know whether you're the tens digit or the ones digit.
5. Ankit: I don't know whether you're the tens digit or the ones digit.

What was Clark's number?

2018.12 Let $f : [0, 1] \to \mathbb{R}$ be a monotonically increasing function such that

$$f\left(\frac{x}{3}\right) = \frac{f(x)}{2}$$

$$f(10x) = 2018 - f(x).$$

If $f(1) = 2018$, find $f\left(\frac{12}{13}\right)$.

2018.13 Find the value of

$$\frac{1}{\sqrt{2}^1} + \frac{4}{\sqrt{2}^2} + \frac{9}{\sqrt{2}^3} + \cdots$$

2018.14 Let $F_1 = 0$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Compute

$$\sum_{n=1}^{\infty} \frac{\sum_{n=1}^{\infty} F_i}{3^n}.$$

2018.15 Let triangle ABC have side lengths $AB = 13$, $BC = 14$, $AC = 15$. Let I be the incenter of ABC. The circle centered at A of radius AI intersects the circumcircle of ABC at H and J. Let L be a point that lies on both the incircle of ABC and line HJ. If the minimal possible value of AL is $\sqrt{n}$, where $n \in \mathbb{Z}$, find n.

2019.1 Find the maximum integral value of k such that $0 \le k \le 2019$ and $|e^{2\pi i \frac{k}{2019}} - 1|$ is maximal.

2019.2 Find the remainder when 2^{2019} is divided by 7.

2019.3 A cylinder with radius 5 and height 1 is rolling on the (unslanted) floor. Inside the cylinder, there is water that has constant height $\frac{15}{2}$ as the cylinder rolls on the floor. What is the volume of the water?

2019.4 Let C be the number of ways to arrange the letters of the word CATALYSIS, T be the number of ways to arrange the letters of the word TRANSPORT, S be the number of ways to arrange the letters of the word STRUCTURE, and M be the number of ways to arrange the letters of the word MOTION. What is $\frac{C-T+S}{M}$?

2019.5 What is the minimum distance between $(2019, 470)$ and $(21a - 19b, 19b + 21a)$ for $a, b \in Z$?

2019.6 At a party, 2019 people decide to form teams of three. To do so, each turn, every person not on a team points to one other person at random. If three people point to each other (that is, A points to B, B points to C, and C points to A), then they form a team. What is the probability that after $65, 536$ turns, exactly one person is not on a team

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2019.7 How many distinct ordered pairs of integers (b, m, t) satisfy the equation $b^8 + m^4 + t^2 + 1 = 2019$?

2019.8 Let (k_i) be a sequence of unique nonzero integers such that $x^2 - 5x + k_i$ has rational solutions. Find the minimum possible value of

$$\frac{1}{5}\sum_{i=1}^{\infty}\frac{1}{k_i}$$

2019.9 You wish to color every vertex, edge, face, and the interior of a cube one color each such that no two adjacent objects are the same color. Faces are adjacent if they share an edge. Edges are adjacent if they share a vertex. The interior is adjacent to all of its faces, edges, and vertices. Each face is adjacent to all of its edges and vertices. Each edge is adjacent to both of its vertices. What is the minimum number of colors required to do this?

2019.10 Compute the remainder when the product of all positive integers less than and relatively prime to 2019 is divided by 2019.

2019.11 A baseball league has 64 people, each with a different 6-digit binary number whose base-10 value ranges from 0 to 63. When any player bats, they do the following: for each pitch, they swing if their corresponding bit number is a 1, otherwise, they decide to wait and let the ball pass. For example, the player with the number 11 has binary number 001011. For the first and second pitch, they wait; for the third, they swing, and so on. Pitchers follow a similar rule to decide whether to throw a splitter or a fastball, if the bit is 0, they will throw a splitter, and if the bit is 1, they will throw a fastball.

If a batter swings at a fastball, then they will score a hit; if they swing on a splitter, they will miss and get a "strike." If a batter waits on a fastball, then they will also get a strike. If a batter waits on a splitter, then they get a "ball." If a batter gets 3 strikes, then they are out; if a batter gets 4 balls, then they automatically get a hit. For example, if player 11 pitched against player 6 (binary is 000110), the batter would get a ball for the first pitch, a ball for the second pitch, a strike for the third pitch, a strike for the fourth pitch, and a hit for the fifth pitch; as a result, they will count that as a "hit." If player 11 pitched against player 5 (binary is 000101), however, then the fifth pitch would be the batter's third strike, so the batter would be "out."

Each player in the league plays against every other player exactly twice; once as batter, and once as pitcher. They are then given a score equal to the number of outs they throw as a pitcher plus the number of hits they get as a batter. What is the highest score received?

2019.12 2019 people (all of whom are perfect logicians), labeled from 1 to 2019, partake in a paintball duel. First, they decide to stand in a circle, in order, so that Person 1 has Person 2 to his left and person 2019 to his right. Then, starting with Person 1 and moving to the left, every person who has not been eliminated takes a turn shooting. On their turn, each person can choose to either shoot one non-eliminated person of his or her choice (which eliminates that person from the game), or deliberately miss. The last person standing wins. If, at any point, play goes around the circle once with no one getting eliminated (that is, if all the people playing decide to miss), then automatic paint sprayers will turn on, and end the game with everyone losing. Each person will, on his or her turn, always pick a move that leads to a win if possible, and, if there is still a choice in what move to make, will prefer shooting over missing, and shooting a person closer to his or her left over shooting someone farther from their left. What is the number of the person who wins this game? Put "0" if no one wins.

2019.13 Triangle $\triangle ABC$ has $AB = 13$, $BC = 14$, and $CA = 15$. $\triangle ABC$ has incircle γ and circumcircle ω. γ has center at I. Line AI is extended to hit ω at P. What is the area of quadrilateral $ABPC$?

2019.5 A regular hexagon has positive integer side length. A laser is emitted from one of the hexagon's corners, and is reflected off the edges of the hexagon until it hits another corner. Let a be the distance that the laser travels. What is the smallest possible value of a^2 such that $a > 2019$?

You need not simplify/compute exponents.

2019.15 A group of aliens from Gliese 667 Cc come to Earth to test the hypothesis that mathematics is indeed a universal language. To do this, they give you the following information about their mathematical system:

• For the purposes of this experiment, the Gliesians have decided to write their equations in the same syntactic format as in Western math. For example, in Western math, the expression "$5+4$" is interpreted as running the "$+$" operation on numbers 5 and 4. Similarly, in Gliesian math, the expression $\alpha\gamma\beta$ is interpreted as running the "γ" operation on numbers α and β.

• You know that γ and η are the symbols for addition and multiplication (which works the same in Gliesian math as in Western math), but you don't know which is which. By some bizarre coincidence, the symbol for equality is the same in Gliesian math as it is in Western math; equality is denoted with an "$=$" symbol between the two equal values.

• Two symbols that look exactly the same have the same meaning. Two symbols that are different have different meanings and, therefore, are not equal.

They then provide you with the following equations, written in Gliesian, which are known to be true:
https://cdn.artofproblemsolving.com/attachments/b/e/e2e44c257830ce8eee7c05535046c17ae3b7e
png