III Iberoamerican Interuniversitary Mathematics Competition - Ecuador

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by Ozc

Problem 1 Find all real numbers a for which there exist different real numbers b, c, d different from a such that the four tangents drawn to the curve $y = \sin(x)$ at the points $(a, \sin(a)), (b, \sin(b)), (c, \sin(c))$ and $(d, \sin(d))$ form a rectangle.

Problem 2 Let k be a positive integer, and let a be an integer such that $a - 2$ is a multiple of 7 and $a^6 - 1$ is a multiple of 7^k.

Prove that $(a+1)^6 - 1$ is also a multiple of 7^k.

Problem 3 Let $f(x)$ be a rational function with complex coefficients whose denominator does not have multiple roots. Let $u_0, u_1, ..., u_n$ be the complex roots of f and $w_1, w_2, ..., w_m$ be the roots of f'. Suppose that u_0 is a simple root of f. Prove that

$$\sum_{k=1}^{m} \frac{1}{w_k - u_0} = 2\sum_{k=1}^{n} \frac{1}{u_k - u_0}.$$

Problem 4 For $n \geq 3$, let $(b_0, b_1, ..., b_{n-1}) = (1, 1, 1, 0, ..., 0)$. Let $C_n = (c_{i,j})$ the $n \times n$ matrix defined by $c_{i,j} = b_{(j-i) \mod n}$. Show
that $\det(C_n) = 3$ if n is not a multiple of 3 and $\det(C_n) = 0$ if n
is a multiple of 3.

Problem 5 Let n be a positive integer with d digits, all different from zero. For $k = 0, ..., d-1$, we define n_k as the number obtained by moving the last k digits of n to the beginning. For example, if $n = 2184$ then $n_0 = 2184, n_1 = 4218, n_2 = 8421, n_3 = 1842$. For m a positive integer, define $s_m(n)$ as the number of values k such that n_k is a multiple of m. Finally, define a_d as the number of integers n with d digits all nonzero, for which $s_2(n) + s_3(n) + s_5(n) = 2d$.
Find

$$\lim_{d\to\infty} \frac{a_d}{5^d}.$$

Problem 6 Let Γ be the branch $x > 0$ of the hyperbola $x^2 - y^2 = 1$. Let $P_0, P_1, ..., P_n$ different points of Γ with $P_0 = (1, 0)$ and $P_1 = (13/12, 5/12)$. Let t_i be the tangent line to Γ at P_i. Suppose that for all $i \geq 0$ the area of the region bounded by t_i, t_{i+1} and Γ is a constant independent of i. Find the coordinates of the points P_i.
