Rice Math Tournament - Team Round, years 2003-09 (years 2003-04, 2006-09, 2012-14 had same problems with Stanford MT)

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2003 **p1.** What is the ratio of the area of an equilateral triangle to the area of the largest rectangle that can be inscribed inside the triangle?

p2. Define

$P(x) = x^{12}+12x^{11}+66x^{10}+220x^9+495x^8+792x^7+924x^6+792x^5-159505x^4+220x^3+66x^2+12x+1.$

Find $\frac{P(19)}{20^4}$.

p3. Four flattened colored cubes are shown below. Each of the cubes' faces has been colored red (R), blue (B), green (G) or yellow (Y). The cubes are stacked on top of each other in numerical order with cube#1 on bottom. The goal of the puzzle is to find an orientation for each cube so that on each of the four visible sides of the stack all four colors appear. Find a solution, and for each side of the stack, list the colors from bottom to top. List the sides in clockwise order.
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`png`

p4. When evaluated, the sum $\sum^{2002}_{k=1}[k \cdot k!]$ is a number that ends with a long series of 9s. How many 9s are at the end of the number?

p5. Find the positive integer n that maximizes the expression $\frac{200003^n}{(n!)^2}$.

p6. Find $11^3 + 12^3 + ... + 100^3$

Hint: Develop a formula for $s(x) =\sum^x_{i=1} i^3$ using perhaps $x^3 - (x - 1)^3$.
Also the formulas $\sum^x_{i=1} i = \frac{x(x+1)}{2}$ and $\sum^x_{i=1} i^2 = \frac{x(x+1)(2x+1)}{6}$ may be helpful.

p7. Six fair 6-sided dice are rolled. What is the probability that the sum of the values on the top faces of the dice is divisible by 7?

p8. Several students take a quiz which has five questions, and each one is worth a point. They are unsure as to how many points they received, but all of them have a reasonable idea about their scores. Below is a table of what each person thinks is the probability that he or she got

each score. Assuming their probabilities are correct, what is the probability that the sum of their scores is exactly 20?
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p9. Let F_n be the number of ways of completely covering an $3 \times n$ chessboard with n 3×1 dominoes. For example, there are two ways of tiling a 3×3 chessboard with three 3×1 dominoes (all horizontal or all vertical). What is F_{14}?

p10. Two players (Kate and Adam) are playing a variant of Nim. There are 11 sticks in front of the players and they take turns each removing either one or any prime number of sticks. The player who is forced to take the last stick loses. The problem with the game is that if player one (Kate) plays perfectly, she will always win. Give the sum of all the starting moves that lead to a sure win for Kate (assuming each player plays perfectly).

p11. Define $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$. Find all (x, y) such that $(f(x, y))^2 - (g(x, y))^2 = \frac12$ and $f(x, y) \cdot g(x, y) = \frac{\sqrt3}{4}$.

p12. The numerals on digital clocks are made up of seven line segments. When various combinations of them light up different numbers are shown. When a digit on the clock changes, some segments turn on and others turn off. For example, when a 4 changes into a 5 two segments turn on and one segment turns off, for a total of 3 changes. In the usual ordering $1, 2, 3, ..., 0$ there are a sum total of 32 segment changes (including the wrapping around from 0 back to 1). If we can put the digits in any order, what is the fewest total segment changes possible? (As above, include the change from the last digit back to the first. Note that a 1 uses the two vertical segments on the right side, a 6 includes the top segment, and a 9 includes the bottom segment.)

p13. How many solutions are there to $(\cos 10x)(\cos 9x) = \frac12$ for $x \in [0, 2\pi]$?

p14. Find $\binom{2003}{0} + \binom{2003}{4} + \binom{2003}{8} + ...$

Hint: Consider $(1 \pm i)$ and (1 ± 1).

p15. Clue is a board game in which four players attempt to solve a mystery. Mr. Boddy has been killed in one of the nine rooms in his house by one of six people with one of six specific weapons. The board game has 21 cards, each with a person, weapon or room on it. At the beginning of each game, one room card, one person card and one weapon card are randomly chosen and set aside as the solution to the mystery. The remaining cards are all gathered, shuffled, and dealt out to the four players. Note that two players get four cards and two get five cards. Players thus start out knowing only that the cards in their hand are not among the solution cards. Essentially, players then take turns guessing the solution to the mystery. A guess is a listing of a person, weapon and a room. Assume that the player who goes first in the game has only 4 cards. What is the probability that his first guess, the first guess of the game, is exactly right (i.e. he guesses all three hidden cards corrrectly?

PS. You had better use hide for answers.

2004 **p1.** Find $\sin x - \cos x$ if $\sin 2x = \frac{2002}{2003}$ and $\frac{5\pi}{4} < x < \frac{9\pi}{4}$.

p2. Dave and Daly decide to play a game using two dice. Dave will roll first, then they will continue alternating turns. If Dave gets a total of exactly 6 before Daly gets a total of exactly 7, then Dave wins. Otherwise Daly wins. What is the probability that Dave wins?

p3. Suppose $ABCDEF$ is a regular hexagon with area 1, and consider the diagonals AC, BD, CE, DF, EA and FB. A star is formed by alternating between the vertices of the hexagon and the points of intersection of the diagonals. What is the area of the star?

p4. A pair of positive integers is golden if they end in the same two digits. For example $(139, 2739)$ and $(350, 850)$ are golden pairs. What is the sum of all two-digit integers n for which (n^2, n^3) is golden?

p5. A polynomial is monic if the leading coefficient is 1. Suppose $p(x)$ is a cubic monic polynomial. Let a, b, and c be the roots of $p(x)$. If $a+b+c = 1$ and $a^2+b^2+c^2 = 5$ and $a^3+b^3+c^3 = 16$, determine $p(x)$.

p6. Thirteen students, four of whom are named Bob, are going out to a math conference. They have three distinct cars, two of which will hold 4 students and the other will hold 5. If the students randomly scramble into the cars, what is the probability that every car has at least one Bob in it?

p7. Find all x such that $\sum_{k=1}^{\infty} kx^k = 20$.

p8. Let $\vartriangle ABC$ be an equilateral triangle with side length 2004. For any point P inside $\vartriangle ABC$, let $d(P)$ be the sum of the distances of P to each of the sides of $\vartriangle ABC$. Let m be the minimum value of $d(P)$. Let M be the maximum value of $d(P)$. Compute $M - m$.

p9. There are 30 owls in a row starting from left to right and Sammy (their coach) is facing them. They have a warmup routine involving trading places in rounds. Each round involves three phases. In the first phase of each round, the owls in the odd positions rotate to the right while the owls in even positions do not move. For example, the first owl moves to the third owl's spot who is moving to the fifth owl's spot, etc. Note the 29th owl will wrap around and move to the first owl's spot. The second phase occurs after the first is completed and in this phase the owls in positions which leave a remainder of 1 when divided by 3 rotate to the right. All other owls stay fixed. For example, the owl in position 1 would move to position 4. In the last phase of each round only owls in positions that leave a remainder of 1 when divided by 5 move. These owls however move left! The next round then begins back at phase 1. The warmup continues until the owls are back in their original order. How many rounds does this take?

p10. Let $\lfloor x \rfloor$ denote the floor function (the largest integer less than or equal to x). Find the average value of the quantity $\lfloor 2x^3 - 2\lfloor x^3 \rfloor \rfloor$ on the interval $\left(-\frac{3}{2}, \frac{3}{2}\right)$.

p11. Suppose a king has $25!$ grains of rice after collecting taxes. He is feeling generous so he instructs his accountant to divide the rice into two piles, the first of which he will keep and the second of which he will distribute to the poor. The accountant places the nth grain of rice in the first pile if n and $25!$ are not relatively prime and the second pile otherwise. What fraction of the grains get distributed to the poor?

p12. Given that $x^2 - 3x + 1 = 0$, find $x^9 + x^7 + x^{-9} + x^{-7}$.

p13. How many ordered triples of integers (b, c, d) are there such that $x^4 - 5x^3 + bx^2 + cx + d$ has four (not necessarily distinct) non-negative rational roots?

p14. Find the length of the leading non-repeating block in the decimal expansion of $\frac{2004}{7*5^{2003}}$. (For example, the length of the leading non-repeating block of $\frac{5}{12} = .41666666\overline{6}$ is 2).

p15. Suppose we play a variant of Yahtzee where we roll 5 dice, trying to get all sixes. After each roll we keep the dice that are sixes and re-roll all the others. What is the probability of getting all sixes in exactly k rolls as a function of k?

PS. You had better use hide for answers.

2005 **p1.** Find the largest prime whose cube divides $1!2!...2005!$.

p2. What is the number of sides of the regular polygon with the largest number of sides whose interior angles measure an integer multiple of 7^o.

p3. A solid is constructed out of an infinite number of cones with height h. The bottom cone has base diameter h. Each successive cone has as its base the circular cross-section halfway up the previous cone. Find the volume of the solid.

p4. Suppose $\triangle ABC$ is a triangle with area 25. Points P, Q and R are on sides AB, BC and CA respectively so that $\frac{BP}{AP} = \frac{CQ}{BQ} = \frac{AR}{CR} = r$. If the area of $\triangle\ PQR$ is 7, what is the sum of all possible values of r?

p5. You are walking up a staircase with stairs that are 1 ft. tall and 1 ft. deep. The number of stairs from the ground floor to the first floor is 1. From the first floor to the second is 2 stairs. From the 99th to the hundredth stair is 100 steps. At each floor, the staircase makes a 90^o turn to the right. At the top of the 100th floor, how far away from the bottom of the staircase are you?

p6. A point in 3-dimensional space is called a lattice point if all three of its coordinates (x, y, z) are integers. When making a list of lattice points a_1, a_2, ... , a_n, what is the minimum n that guarantees the midpoint between some 2 of the lattice points in the list is a lattice point?

p7. Sparc is played with an octahedral and a dodecahedral die, numbered $1-8$ and $1-12$. If a player rolls a sum of 2, 6, 11, or 20 he wins. Of the other possible sums, a casino picks some which cause the player to lose. If the player rolls any of the other sums, they roll repeatedly until they get an 11 or their first roll. If they roll an 11 first they lose, if they roll their first roll, they win. Given that the probabilty of winning is $\frac{23242}{110880}$ and that given a choice between two equal probability rolls, the one with greater sum loses, which sums allow the player to keep rolling?

p8. How many right triangles with integer side lengths have one leg (not the hypotenuse) of length 60 ?

p9. Let S be the set of the first nine positive integers, and let A be a nonempty subset of S. The mirror of A is the set formed by replacing each element m of A by $10-m$. For example, the mirror of $1, 3, 5, 6$ is $4, 5, 7, 9$. A nonempty subset of S is reflective if it is equivalent to its mirror. What is the probability that a randomly chosen nonempty subset of S is reflective?

p10. Approximate to the nearest tenth $\sqrt{2000 \cdot 2010}$.

p11. Each of the small equilateral triangles (9 total) have side length x and is randomly colored red or blue. What is the probability that there will be an equilateral triangle of side length 2x or 3x that is entirely red or entirely blue?
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p12. Craig Fratrick walks from home to a nearby Dunkin Donuts. He walks East a distance of 30 meters. Then he turns 15^o to the left and walks 30 meters. He repeats this process until he has traveled 210 meters and arrives at Dunkin Donuts. If he had walked directly from home to Dunkin Donuts, how much distance could he have saved by walking directly from home to Dunkin Donuts (in one straight line).

p13. Let
$$P = \cos \frac{\pi}{4} \cos \frac{\pi}{8} ... \cos \frac{\pi}{2^{1000}}.$$
What is $2\pi P$ to the nearest integer?

p14. For all real numbers x, let the mapping $f(x) = \frac{1}{x-i}$. There are real numbers a, b, c and d for which $f(a)$, $f(b)$, $f(c)$, and $f(d)$ form a square in the complex plane. What is the area of the square?

p15. The Fibonacci numbers are defined recursively so that $F_0 = 0$, $F_1 = 1$, and for $n > 1$, $F_n = F_{n-1} + F_{n-2}$. Calculate $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n \cdot F_n$.

PS. You had better use hide for answers.

2006.1 Given $\triangle ABC$, where A is at $(0, 0)$, B is at $(20, 0)$, and C is on the positive y-axis. Cone M is formed when $\triangle ABC$ is rotated about the x-axis, and cone N is formed when $\triangle ABC$ is rotated about the y-axis. If the volume of cone M minus the volume of cone N is 140π, find the length of $\overline{BC}$.

2006.2 In a given sequence $\{S_1, S_2, ..., S_k\}$, for terms $n \ge 3$, $S_n = \sum_{i=1}^{n-1} i \cdot S_{n-i}$. For example, if the first two elements are 2 and 3, respectively, the third entry would be $1 \cdot 3 + 2 \cdot 2 = 7$, and the fourth would be $1 \cdot 7 + 2 \cdot 3 + 3 \cdot 2 = 19$, and so on. Given that a sequence of integers having this form starts with 2, and the 7th element is 68, what is the second element?

2006.3 A triangle has altitudes of length 5 and 7. What is the maximum length of the third altitude?

2006.4 Let $x + y = a$ and $xy = b$. The expression $x^6 + y^6$ can be written as a polynomial in terms of a and b. What is this polynomial?

2006.5 There exist two positive numbers x such that $\sin(\arccos(\tan(\arcsin x))) = x$. Find the product of the two possible x.

2006.6 The expression $16^n + 4^n + 1$ is equiavalent to the expression $(2^{p(n)} - 1)/(2^{q(n)} - 1)$ for all positive integers $n > 1$ where $p(n)$ and $q(n)$ are functions and $\frac{p(n)}{q(n)}$ is constant. Find $p(2006) - q(2006)$.

2006.7 Let S be the set of all 3-tuples (a, b, c) that satisfy $a + b + c = 3000$ and $a, b, c > 0$. If one of these 3-tuples is chosen at random, what's the probability that a, b or c is greater than or equal to 2,500?

2006.8 Evaluate: $\lim_{n\to\infty} \sum_{k=n^2}^{(n+1)^2} \dfrac{1}{\sqrt{k}}$

2006.9 $\triangle ABC$ has $AB = AC$. Points M and N are midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. The medians $\overline{MC}$ and $\overline{NB}$ intersect at a right angle. Find $(\frac{AB}{BC})^2$.

2006.10 Find the smallest positive m for which there are at least 11 even and 11 odd positive integers n so that $\frac{n^3+m}{n+2}$ is an integer.

2006.11 Polynomial $P(x) = c_{2006}x^{2006} + c_{2005}x^{2005} + \ldots + c_1 x + c_0$ has roots $r_1, r_2, \ldots, r_{2006}$. The coefficients satisfy $2i\frac{c_i}{c_{2006-i}} = 2j\frac{c_j}{c_{2006-j}}$ for all pairs of integers $0 \le i, j \le 2006$. Given that $\sum_{i\ne j, i=1, j=1}^{2006} \frac{r_i}{r_j} = 42$, determine $\sum_{i=1}^{2006}(r_1 + r_2 + \ldots + r_{2006})$.

2006.12 Find the total number of k-tuples $(n_1, n_2, ..., n_k)$ of positive integers so that $n_{i+1} \ge n_i$ for each i, and k regular polygons with numbers of sides $n_1, n_2, ..., n_k$ respectively will fit into a tesselation at a point. That is, the sum of one interior angle from each of the polygons is 360°.

2006.13 A ray is drawn from the origin tangent to the graph of the upper part of the hyperbola $y^2 = x^2 - x + 1$ in the first quadrant. This ray makes an angle of θ with the positive x-axis. Compute $\cos\theta$.

2006.14 Find the smallest nonnegative integer n for which $\binom{2006}{n}$ is divisible by 7^3.

2006.15 Let c_i denote the ith composite integer so that $\{c_i\} = 4, 6, 8, 9, ...$ Compute

$$\prod_{i=1}^{\infty} \frac{c_i^2}{c_i^2 - 1}$$

(Hint: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$)

2007.1 How many rational solutions for x are there to the equation $x^4 + (2-p)x^3 + (2-2p)x^2 + (1-2p)x - p = 0$ if p is a prime number?

2007.2 If a and b are each randomly and independently chosen in the interval $[-1, 1]$, what is the probability that $|a| + |b| < 1$?

2007.3 A clock currently shows the time $10 : 10$. The obtuse angle between the hands measures x degrees. What is the next time that the angle between the hands will be x degrees? Round your answer to the nearest minute.

2007.4 What is the area of the smallest triangle with all side lengths rational and all vertices lattice points?

2007.5 How many five-letter "words" can you spell using the letters S, I, and T, if a "word" is defines as any sequence of letters that does not contain three consecutive consonants?

2007.6 $x \equiv \left(\sum_{k=1}^{2007} k\right) \mod 2016$, where $0 \le x \le 2015$. Solve for x.

2007.7 Daniel counts the number of ways he can form a positive integer using the digits $1, 2, 2, 3$, and 4 (each digit at most once). Edward counts the number of ways you can use the letters in the word "$BANANAS$" to form a six-letter word (it doesn't have to be a real English word). Fernando counts the number of ways you can distribute nine identical pieces of candy to three children. By their powers combined, they add each of their values to form the number that represents the meaning of life. What is the meaning of life? (Hint: it's not 42.)

2007.8 A 13-foot tall extraterrestrial is standing on a very small spherical planet with radius 156 feet. It sees an ant crawling along the horizon. If the ant circles the extraterrestrial once, always staying on the horizon, how far will it travel (in feet)?

2007.9 Let d_n denote the number of derangements of the integers $1, 2, \ldots n$ so that no integer i is in the i^{th} position. It is possible to write a recurrence relation $d_n = f(n)d_{n-1} + g(n)d_{n-2}$; what is $f(n) + g(n)$?

2007.10 A nondegenerate rhombus has side length l, and its area is twice that of its inscribed circle. Find the radius of the inscribed circle.

2007.11 The polynomial $R(x)$ is the remainder upon dividing x^{2007} by x^2-5x+6. $R(0)$ can be expressed as $ab(a^c - b^c)$. Find $a + c - b$.

2007.12 Brownian motion (for example, pollen grains in water randomly pushed by collisions from water molecules) simplified to one dimension and beginning at the origin has several interesting properties. If $B(t)$ denotes the position of the particle at time t, the average of $B(t)$ is $x = 0$, but the averate of $B(t)^2$ is t, and these properties of course still hold if we move the space and time origins ($x = 0$ and $t = 0$) to a later position and time of the particle (past and future are independent). What is the average of the product $B(t)B(s)$?

2007.13 Mary Jane and Rachel are playing ping pong. Rachel has a $7/8$ chance of returning any shot, and Mary Jane has a $5/8$ chance. Mary Jane serves to Rachel (and doesn't mess up the serve).

What is the average number of returns made?

2007.14 Let p, q be positive integers and let $x_0 = 0$. Suppose $x_{n+1} = x_n + p + \sqrt{q^2 + 4px_n}$. Find an explicit formula for x_n.

2007.15 Evaluate $\int_0^\infty \frac{\tan^{-1}(\pi x)-\tan^{-1} x}{x} dx$

2008 **p1**. Find the maximum value of $e^{\sin x \cos x \tan x}$.

p2. A fighter pilot finds that the average number of enemy ZIG planes she shoots down is $56z - 4z^2$, where z is the number of missiles she fires. Intending to maximize the number of planes she shoots down, she orders her gunner to "Have a nap ... then fire z missiles!" where z is an integer. What should z be?

p3. A sequence is generated as follows: if the n^{th} term is even, then the $(n + 1)^{th}$ term is half the n^{th} term; otherwise it is two more than twice the n^{th} term. If the first term is 10, what is the 2008^{th} term?

p4. Find the volume of the solid formed by rotating the area under the graph of $y = \sqrt{x}$ around the x-axis, with $0 \le x \le 2$.

p5. Find the volume of a regular octahedron whose vertices are at the centers of the faces of a unit cube.

p6. What is the area of the triangle with vertices $(x, 0, 0)$, $(0, y, 0)$, and $(0, 0, z)$?

p7. Daphne is in a maze of tunnels shown below. She enters at A, and at each intersection, chooses a direction randomly (including possibly turning around). Once Daphne reaches an exit, she will not return into the tunnels. What is the probability that she will exit at A?
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`png`

p8. In triangle AXE, T is the midpoint of $\overline{EX}$, and P is the midpoint of $\overline{ET}$. If triangle APE is equilateral, find $\cos(m\angle XAE)$.

p9. In rectangle $XKCD$, J lies on $\overline{KC}$ and Z lies on $\overline{XK}$. If $\overline{XJ}$ and $\overline{KD}$ intersect at Q, $\overline{QZ} \perp \overline{XK}$, and $\frac{KC}{KJ} = n$, find $\frac{XD}{QZ}$.

p10. Bill the magician has cards A, B, and C as shown. For his act, he asks a volunteer to pick any number from 1 through 8 and tell him which cards among A, B, and C contain it. He then uses this information to guess the volunteer's number (for example, if the volunteer told Bill "A and C", he would guess "3").
One day, Bill loses card C and cannot remember which numbers were on it. He is in a hurry and randomly chooses four different numbers from 1 to 8 to write on a card. What is the probability Bill will still be able to do his trick?

A: 2 3 5 7

B: 2 4 6 7

C: 2 3 6 1

p11. Given that $f(x, y) = x^7y^8 + x^4y^{14} + A$ has root $(16, 7)$, find another root.

p12. How many nonrectangular trapezoids can be formed from the vertices of a regular octagon?

p13. If $re^{i\theta}$ is a root of $x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0$, $r > 0$, and $0 \le \theta < 360$ with θ in degrees, find all possible values of θ.

p14. For what real values of n is $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\tan(x))^n dx$ defined?

p15. A parametric graph is given by

$$\begin{cases} y = \sin t \\ x = \cos t + \frac{1}{2}t \end{cases}$$

How many times does the graph intersect itself between $x = 1$ and $x = 40$?

PS. You had better use hide for answers.

2009.1 In the future, each country in the world produces its Olympic athletes via cloning and strict training
programs. Therefore, in the fi nals of the 200 m free, there are two indistinguishable athletes from each
of the four countries. How many ways are there to arrange them into eight lanes?

2009.2 Factor completely the expression $(a - b)^3 + (b - c)^3 + (c - a)^3$

2009.3 If x and y are positive integers, and $x^4 + y^4 = 4721$, find all possible values of $x + y$

2009.4 How many ways are there to write 657 as a sum of powers of two where each power of two is used at
most twice in the sum? For example, $256 + 256 + 128 + 16 + 1$ is a valid sum.

2009.5 Compute $\int_0^\infty t^5 e^{-t} dt$

2009.6 Rhombus $ABCD$ has side length 1. The size of $\angle A$ (in degrees) is randomly selected from all real numbers between 0 and 90. Find the expected value of the area of $ABCD$.

2009.7 An isosceles trapezoid has legs and shorter base of length 1. Find the maximum possible value of its area

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2009.8 Simplify $\sum_{k=1}^n \frac{k^2(k-n)}{n^4}$

2009.9 Find the shortest distance between the point $(6, 12)$ and the parabola given by the equation $x = \frac{y^2}{2}$

2009.10 Evaluate $\sum_{n=2009}^\infty \frac{\binom{n}{2009}}{2^n}$

2009.11 Let z_1 and z_2 be the zeros of the polynomial $f(x) = x^2 + 6x + 11$. Compute $(1 + z_1^2 z_2)(1 + z_1 z_2^2)$.

2009.12 A number N has 2009 positive factors. What is the maximum number of positive factors that N^2 could have?

2009.13 Find the remainder obtained when 17^{289} is divided by 7

2009.14 Let a and b be integer solutions to $17a + 6b = 13$. What is the smallest possible positive value for $a - b$?

2009.15 What is the largest integer n for which $\frac{2008!}{31^n}$ is an integer?

2010 **p1.** Compute $\lim_{x\to 0} \frac{\tan x - x - \frac{x^3}{3}}{x}$.

p2. For how many integers n is $\frac{n}{20-n}$ equal to the square of a positive integer.

p3. Find all possible solutions $(x_1, x_2, ..., x_n)$ to the following equations, where $x_1 = \frac{1}{2}\left(x_n + \frac{x_{n-1}^2}{x_n}\right)$

$x_2 = \frac{1}{2}\left(x_1 + \frac{x_n^2}{x_1}\right) x_3 = \frac{1}{2}\left(x_2 + \frac{x_1^2}{x_2}\right) x_4 = \frac{1}{2}\left(x_3 + \frac{x_2^2}{x_3}\right) x_5 = \frac{1}{2}\left(x_4 + \frac{x_3^2}{x_4}\right) ... x_n = \frac{1}{2}\left(x_{n-1} + \frac{x_n^2}{x_1}\right) = 2010.$

p4. There is an economic crisis in Hogwarts. To generate more money for themselves, Fred and George decided to run a magical reaction that will turn spiders into galleons. In a 100 L cauldron, the initially put in 50 moles of spiders. In addition, they bewitched their brother Ron to continuously feed in spiders at a rate of 1 mol/min. Assume spiders turn into galleons at a constant at a constant rate of 0.5 mol per minute per liter. Also, assume that the reaction occurs uniformly, that Fred is very assiduous in his stirring job, and there are no other reactions. Find the number of spiders remaining after1 hour.

p5. Rank the following in decreasing order: $A = \frac{1\sqrt{1}+2\sqrt{2}+3\sqrt{3}}{\sqrt{1}+\sqrt{2}+\sqrt{3}}$, $B = \frac{1^2+2^2+3^2}{1+2+3}$, $C = \frac{1+2+3}{3}$,$D = \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}}{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}}$.

p6. How many paths are there from A to B in the directed graph below?
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jpg

p7. $\triangle ABC$ is a triangle with $AB = 5$, $BC = 6$, and $CA = 7$. Squares are drawn on each side, as in the image below. Find the area of hexagon $DEFGHI$.
https://cdn.artofproblemsolving.com/attachments/a/0/8fa76ae93a79b367f5ce3af98ad8f7270ea4c
png

p8. Suppose that for an infinitely differentiable function f, $\lim_{x\to 0} \frac{f(4x)+af(3x)+bf(2x)+cf(x)+df(0)}{x^4}$ exists. Find $1000a + 100b + 10c + d$.

p9. Let P be the vertex of a right circular cone (vertex angle is 45^o), C be the center of the circular base, and K be a point on the outline of the base (i.e. the circle that outlines the base) such that the segment $\overline{CK}$ is parallel to the z axis, as show in the diagram below. This cone has its principle axis (PC) rotated 45^o from the x axis in the xy plane. What is the angle between the x axis and the segment PK in radians?
https://cdn.artofproblemsolving.com/attachments/1/2/5f91f77d8bf6e5264c6fedc55f41e81e2c960
png

p10. Positive real numbers x, y, and z satisfy the equations $x^2 + y^2 = 9$ $y^2 + \sqrt{2}yz + z^2 = 16$ $z^2 + \sqrt{2}zx + x^2 = 25$.
Compute $\sqrt{2}xy + yz + zx$.

p11. Find the volume of the region given by the inequality

$$|x + y + z| + |x + y - z| + |x - y + z| + | - x + y + z| \le 4.$$

p12. Suppose we have a polyhedron consisting of triangles and quadrilaterals, and each vertex is shared by exactly 4 triangles and one quadrilateral. How many vertices are there?

p13. A one-dimensional ladder of length c is restricted so that one endpoint must lie on the positive x axis and the other endpoint on the positive y axis. Let f be the curve that is tangent to each of the possible arrangements of ladders. Find an equation for f.

p14. A, B, C, D are points along a circle, in that order. AC intersects BD at X. If $BC = 6$, $BX = 4$, $XD = 5$, and $AC = 11$, find AB.

p15. There are five balls that look identical, but their weights all differ by a little. We have a balance that can compare only two balls at a time. What is the minimum number of times we have to use to balance to rank all balls by weight?

Note: 8 problems were common with 2010 Stanford Math Tournament (`https://artofproblemsolving.com/community/c4h2764579p24200754`): p1-3, p5, p8, p10-12

PS. You had better use hide for answers.

2011 *The 15 problems p1-15 come from Stanford Math Tournament. Rice Math Tournament had 13 same problems and p16, p17 as substitutes of problems p5 and p11.*

p1. Let $ABCD$ be a unit square. The point E lies on BC and F lies onAD. $\triangle AEF$ is equilateral. $GHIJ$ is a square inscribed in $\triangle AEF$ so that $\overline{GH}$ is on $\overline{EF}$. Compute the area of $GHIJ$.
`https://cdn.artofproblemsolving.com/attachments/e/1/e7c02a8c2bee27558a441e4acc9b639f084c7`
`png`

p2. Find all integers x for which $|x^3 + 6x^2 + 2x - 6|$ is prime.

p3. Let A be the set of points (a, b) with $2 < a < 6$, $-2 < b < 2$ such that the equation

$$ax^4 + 2x^3 - 2(2b - a)x^2 + 2x + a = 0$$

has at least one real root. Determine the area of A.

p4. Three nonnegative reals x, y, z satisfy $x + y + z = 12$ and $xy + yz + zx = 21$. Find the maximum of xyz.

p5. Let $\triangle ABC$ be equilateral. Two points D and E are on side BC (with order B, D, E, C), and satisfy $\angle DAE = 30^o$. If $BD = 2$ and $CE = 3$, what is BC?
https://cdn.artofproblemsolving.com/attachments/a/6/4dd9249411e35efaa220b184e2dc3428493b7
png

p6. Three numbers are chosen at random between 0 and 2. What is the probability that the difference between the greatest and least is less than $1/4$?

p7. Tony the mouse starts in the top left corner of a 3×3 grid. After each second, he randomly moves to an adjacent square with equal probability. What is the probability he reaches the cheese in the bottom right corner before he reaches the mousetrap in the center?

p8. Let $A = (0, 0)$, $B = (1, 0)$, and $C = (0, 1)$. Divide AB into n equal segments, and call the endpoints of these segments $A = B_0, B_1, B_2, ..., B_n = B$. Similarly, divide AC into n equal segments with endpoints $A = C_0, C_1, C_2, ..., C_n = C$. By connecting B_i and C_{n-i} for all $0 \le i \le n$, one gets a piecewise curve consisting of the uppermost line segments. Find the equation of the limit of this piecewise curve as n goes to infinity.
https://cdn.artofproblemsolving.com/attachments/7/d/903a6dc58d6f49be75c7aa8fecfdc863c2a74
png

p9. Determine the maximum number of distinct regions into which 2011 circles of arbitrary size can partition the plane.

p10. For positive reals x, y, and z, compute the maximum possible value of $\frac{xyz(x+y+z)}{(x+y)^2(y+z)^2}$.

p11. Find the diameter of an icosahedron with side length 1 (an icosahedron is a regular polyhedron with 20 identical equilateral triangle faces, a picture is given below).
https://cdn.artofproblemsolving.com/attachments/2/d/e5714078e9da58557cde03ae5c93644877669
png

p12. Find the boundary of the projection of the sphere $x^2 + y^2 + (z - 1)^2 = 1$ onto the plane $z = 0$ with respect to the point $P = (0, -1, 2)$. Express your answer in the form $f(x, y) = 0$, where $f(x, y)$ is a function of x and y.

p13. Compute the number of pairs of 2011-tuples $(x_1, x_2, ..., x_{2011})$ and $(y_1, y_2, ..., y_{2011})$ such that $x_k = x^2_{k-1} - y^2_{k-1} - 2$ and $y_k = 2x_{k-1}y_{k-1}$ for $1 \le k \le 2010$, $x_1 = x^2_{2011} - y^2_{2011} - 2$, and $y_1 = 2x_{2011}y_{2011}$.

p14. Compute $I = \int_0^1 \frac{ln(x+1)}{x^2+1}dx$.

p15. Find the smallest $\alpha > 0$ such that there exists $m > 0$ making the following equation hold for all positive integers $a, b \ge 2$:

$$\left(\frac{1}{gcd(a, b-1)} + \frac{1}{gcd(a-1, b)}\right)(a+b)^\alpha \ge m.$$

Rice Math Tournament problems (substitutes of problems p5, and p11).

p16. If $f(x) = (x-1)^4(x-2)^3(x-3)^2$, find $f'''(1) + f''(2) + f'(3)$.

p17. Find the unique polynomial $P(x)$ with coefficients taken from the set $\{-1, 0, 1\}$ and with least possible degree such that $P(2010) \equiv 1$ (mod 3), $P(2011) \equiv 0$ (mod 3), and $P(2012) \equiv 0$ (mod 3).

PS. You had better use hide for answers.

2012 **p1.** How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ take on exactly 3 distinct values?

p2. Let i be one of the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$. Suppose that for all positive integers n, the number n^n never has remainder i upon division by 12. List all possible values of i.

p3. A card is an ordered 4-tuple (a_1, a_2, a_3, a_4) where each a_i is chosen from $\{0, 1, 2\}$. A line is an (unordered) set of three (distinct) cards $\{(a_1, a_2, a_3, a_4)$,(b_1, b_2, b_3, b_4),$(c_1, c_2, c_3, c_4)\}$ such that for each i, the numbers a_i, b_i, c_i are either all the same or all different. How many different lines are there?

p4. We say that the pair of positive integers (x, y), where $x < y$, is a k-tangent pair if we have $\arctan\frac{1}{k} = arctan\frac{1}{x} + arctan\frac{1}{y}$. Compute the second largest integer that appears in a 2012-tangent pair.

p5. Regular hexagon $A_1A_2A_3A_4A_5A_6$ has side length 1. For $i = 1, ..., 6$, choose B_i to be a point on the segment A_iA_{i+1} uniformly at random, assuming the convention that $A_{j+6} = A_j$ for all integers j. What is the expected value of the area of hexagon $B_1B_2B_3B_4B_5B_6$?

p6. Evaluate $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm(n+m+1)}$.

p7. A plane in 3-dimensional space passes through the point (a_1, a_2, a_3), with a_1, a_2, and a_3 all positive. The plane also intersects all three coordinate axes with intercepts greater than zero (i.e. there exist positive numbers b_1, b_2, b_3 such that $(b_1, 0, 0)$, $(0, b_2, 0)$, and $(0, 0, b_3)$ all lie on this plane). Find, in terms of a_1, a_2, a_3, the minimum possible volume of the tetrahedron formed by the origin and these three intercepts.

p8. The left end of a rubber band e meters long is attached to a wall and a slightly sadistic child holds on to the right end. A point-sized ant is located at the left end of the rubber band at time $t = 0$, when it begins walking to the right along the rubber band as the child begins stretching it. The increasingly tired ant walks at a rate of $1/(ln(t + e))$ centimeters per second, while the child uniformly stretches the rubber band at a rate of one meter per second. The rubber band is infinitely stretchable and the ant and child are immortal. Compute the time in seconds, if it exists, at which the ant reaches the right end of the rubber band. If the ant never reaches the right end, answer $+\infty$.

p9. We say that two lattice points are neighboring if the distance between them is 1. We say that a point lies at distance d from a line segment if d is the minimum distance between the point and any point on the line segment. Finally, we say that a lattice point A is nearby a line segment if the distance between A and the line segment is no greater than the distance between the line segment and any neighbor of A. Find the number of lattice points that are nearby the line segment connecting the origin and the point $(1984, 2012)$.

p10. A permutation of the first n positive integers is valid if, for all $i > 1$, i comes after $\left\lfloor \frac{i}{2} \right\rfloor$ in the permutation. What is the probability that a random permutation of the first 14 integers is valid?

p11. Given that $x, y, z > 0$ and $xyz = 1$, find the range of all possible values of $\frac{x^3+y^3+z^3-x^{-3}-y^{-3}-z^{-3}}{x+y+z-x^{-1}-y^{-1}-z^{-1}}$.

p12. A triangle has sides of length $\sqrt{2}$, $3 + \sqrt{3}$, and $2\sqrt{2} + \sqrt{6}$. Compute the area of the smallest regular polygon that has three vertices coinciding with the vertices of the given triangle.

p13. How many positive integers n are there such that for any natural numbers a, b, we have $n|(a^2b + 1)$ implies $n|(a^2 + b)$?

p14. Find constants a and c such that the following limit is finite and nonzero: $c = \lim_{n\to\infty} \frac{e\left(1-\frac{1}{n}\right)^n - 1}{n^a}$. Give your answer in the form (a, c).

p15. Sean thinks packing is hard, so he decides to do math instead. He has a rectangular sheet that he wants to fold so that it fits in a given rectangular box. He is curious to know what the optimal size of a rectangular sheet is so that it's expected to fit well in any given box. Let a and b be positive reals with $a \leq b$, and let m and n be independently and uniformly distributed random variables in the interval $(0, a)$. For the ordered 4-tuple (a, b, m, n), let $f(a, b, m, n)$ denote the ratio between the area of a sheet with dimension a×b and the area of the horizontal cross-section of the box with dimension $m \times n$ after the sheet has been folded in halves along each dimension until it occupies the largest possible area that will still fit in the box (because Sean is picky, the sheet must be placed with sides parallel to the box's sides). Compute the smallest value of b/a that maximizes the expectation f.

PS. You had better use hide for answers.

2013.1 Let $f_1(n)$ be the number of divisors that n has, and define $f_k(n) = f_1(f_{k-1}(n))$. Compute the smallest integer k such that $f_k(2013^{2013}) = 2$.

2013.2 In unit square $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. Let M be the midpoint of $\overline{CD}$, with $\overline{AM}$ intersecting $\overline{BD}$ at F and $\overline{BM}$ intersecting $\overline{AC}$ at G. Find the area of quadrilateral $MFEG$.

2013.3 Nine people are practicing the triangle dance, which is a dance that requires a group of three people. During each round of practice, the nine people split off into three groups of three people each, and each group practices independently. Two rounds of practice are different if there exists some person who does not dance with the same pair in both rounds. How many different rounds of practice can take place?

2013.4 For some positive integers a and b, $(x^a + abx^{a-1} + 13)^b(x^3 + 3bx^2 + 37)^a = x^{42} + 126x^{41} + \cdots$. Find the ordered pair (a, b).

2013.5 A polygonal prism is made from a flexible material such that the two bases are regular 2^n-gons $(n > 1)$ of the same size. The prism is bent to join the two bases together without twisting, giving a figure with 2^n faces. The prism is then repeatedly twisted so that each edge of one base becomes aligned with each edge of the base exactly once. For an arbitrary n, what is the sum of the number of faces over all of these configurations (including the non-twisted case)?

2013.6 How many distinct sets of 5 distinct positive integers A satisfy the property that for any positive integer $x \leq 29$, a subset of A sums to x?

2013.7 Find all real values of u such that the curves $y = x^2 + u$ and $y = \sqrt{x - u}$ intersect in exactly one point.

2013.8 Rational Man and Irrational Man both buy new cars, and they decide to drive around two racetracks from time $t = 0$ to time $t = \infty$. Rational Man drives along the path parametrized by

$$x = \cos(t)$$
$$y = \sin(t)$$

and Irrational Man drives along the path parametrized by

$$x = 1 + 4\cos\frac{t}{\sqrt{2}}$$
$$y = 2\sin\frac{t}{\sqrt{2}}.$$

Find the largest real number d such that at any time t, the distance between Rational Man and Irrational Man is not less than d.

2013.9 Charles is playing a variant of Sudoku. To each lattice point (x, y) where $1 \le x, y < 100$, he assigns an integer between 1 and 100 inclusive. These integers satisfy the property that in any row where $y = k$, the 99 values are distinct and never equal to k; similarly for any column where $x = k$. Now, Charles randomly selects one of his lattice points with probability proportional to the integer value he assigned to it. Compute the expected value of $x + y$ for the chosen point (x, y).

2013.10 A unit circle is centered at the origin and a tangent line to the circle is constructed in the first quadrant such that it makes an angle $5\pi/6$ with the y-axis. A series of circles centered on the x-axis are constructed such that each circle is both tangent to the previous circle and the original tangent line. Find the total area of the series of circles.

2013.11 What is the smalles positive integer with exactly 768 divisors? Your answer may be written in its prime factorization.

2013.12 Suppose Robin and Eddy walk along a circular path with radius r in the same direction. Robin makes a revolution around the circular path every 3 minutes and Eddy makes a revolution every minute. Jack stands still at a distance $R > r$ from the center of the circular path. At time $t = 0$, Robin and Eddy are at the same point on the path, and Jack, Robin, and Eddy, and the center of the path are collinear. When is the next time the three people (but not necessarily the center of the path) are collinear?

2013.13 A board has 2, 4, and 6 written on it. A person repeatedly selects (not necessarily distinct) values for x, y, and z from the board, and writes down $xyz + xy + yz + zx + x + y + z$ if and only if that number is not yet on the board and is also less than or equal to 2013. This person repeats this process until no more numbers can be written. How many numbers will be written at the end of the process?

2013.14 You have a 2 meter long string. You choose a point along the string uniformly at random and make a cut. You discard the shorter section. If you still have 0.5 meters or more of string, you repeat. You stop once you have less than 0.5 meters of string. On average, how many cuts will you make before stopping?

2013.15 Suppose we climb a mountain that is a cone with radius 100 and height 4. We start at the bottom of the mountain (on the perimeter of the base of the cone), and our destination is the opposite side of the mountain, halfway up (height $z = 2$). Our climbing speed starts at $v_0 = 2$ but gets slower at a rate inversely proportional to the distance to the mountain top (so at height z the speed v is $(h - z)v_0/h$). Find the minimum time needed to get to the destination.

2014 **p1.** Given that the three points where the parabola $y = bx^2 - 2$ intersects the x-axis and y-axis form an equilateral triangle, compute b.

p2. Compute the last digit of
$$2^{(3^{(4^{\cdots 2014)})})}$$

p3. A math tournament has a test which contains 10 questions, each of which come from one of three different subjects. The subject of each question is chosen uniformly at random from the three subjects, and independently of the subjects of all the other questions. The test is unfair if any one subject appears at least 5 times. Compute the probability that the test is unfair.

p4. Let S_n be the sum $S_n = 1 + 11 + 111 + 1111 + ... + 111...11$ where the last number $111...11$ has exactly n 1's. Find $\lfloor 10^{2017}/S_{2014} \rfloor$.

p5. ABC is an equilateral triangle with side length 12. Let O_A be the point inside ABC that is equidistant from B and C and is $\sqrt{3}$ units from A. Define O_B and O_C symmetrically. Find the area of the intersection of triangles O_ABC, AO_BC, and ABO_C.

p6. A *composition* of a natural number n is a way of writing it as a sum of natural numbers, such as $3 = 1 + 2$. Let $P(n)$ denote the sum over all compositions of n of the number of terms in the composition. For example, the compositions of 3 are 3, $1+2$, $2+1$, and $1+1+1$; the first has one term, the second and third have two each, and the last has 3 terms, so $P(3) = 1+2+2+3 = 8$. Compute $P(9)$.

p7. Let ABC be a triangle with $AB = 7$, $AC = 8$, and $BC = 9$. Let the angle bisector of A intersect BC at D. Let E be the foot of the perpendicular from C to line AD. Let M be the midpoint of BC. Find ME.

p8. Call a function g *lower-approximating* for f on the interval $[a, b]$ if for all $x \in [a, b]$, $f(x) \geq g(x)$. Find the maximum possible value of $\int_1^2 g(x)dx$ where $g(x)$ is a linear lower-approximating function for $f(x) = x^x$ on $[1, 2]$.

p9. Determine the smallest positive integer x such that $1.24x$ is the same number as the number obtained by taking the first (leftmost) digit of x and moving it to be the last (rightmost) digit of x.

p10. Let a and b be real numbers chosen uniformly and independently at random from the interval $[-10, 10]$. Find the probability that the polynomial $x^5 + ax + b$ has exactly one real root (ignoring multiplicity).

p11. Let b be a positive real number, and let a_n be the sequence of real numbers defined by $a_1 = a_2 = a_3 = 1$, and $a_n = a_{n-1} + a_{n-2} + ba_{n-3}$ for all $n > 3$. Find the smallest value of b such that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}$ diverges.

p12. Find the smallest L such that

$$\left(1 - \frac{1}{a}\right)^b \left(1 - \frac{1}{2b}\right)^c \left(1 - \frac{1}{3c}\right)^a \leq L$$

for all real numbers a, b, and c greater than 1.

p13. Find the number of distinct ways in which $30^{(30^{30})}$ can be written in the form $a^{(b^c)}$, where a, b, and c are integers greater than 1.

p14. Convex quadrilateral $ABCD$ has sidelengths $AB = 7$, $BC = 9$, $CD = 15$. A circle with center I lies inside the quadrilateral, and is tangent to all four of its sides. Let M and N be the midpoints of AC and BD, respectively. It can be proven that I always lies on segment MN. If I is in fact the midpoint of MN, find the area of quadrilateral $ABCD$.

p15. Marc has a bag containing 10 balls, each with a different color. He draws out two balls uniformly at random and then paints the first ball he drew to match the color of the second ball. Then he places both balls back in the bag. He repeats until all the balls are the same color. Compute the expected number of times Marc has to perform this procedure before all the balls are the same color.

PS. You had better use hide for answers.

2015 **p1.** Two externally tangent unit circles are constructed inside square $ABCD$, one tangent to AB and AD, the other to BC and CD. Compute the length of AB.

p2. We say that a triple of integers (a, b, c) is sorted if $a < b < c$. How many sorted triples of positive integers are there such that $c \le 15$ and the greatest common divisor of a, b, and c is greater than 1?

p3. Two players play a game where they alternate taking a positive integer N and decreasing it by some divisor n of N such that $n < N$. For example, if one player is given $N = 15$, they can choose $n = 3$ and give the other player $N - n = 15 - 3 = 12$. A player loses if they are given $N = 1$.
For how many of the first 2015 positive integers is the player who moves first guaranteed to win, given optimal play from both players?

p4. The polynomial $x^3 - 2015x^2 + mx + n$ has integer coefficients and has 3 distinct positive integer roots. One of the roots is the product of the two other roots. How many possible values are there for n?

p5. You have a robot. Each morning the robot performs one of four actions, each with probability $1/4$: • Nothing. • Self-destruct. • Create one clone. • Create two clones.
Compute the probability that you eventually have no robots.

p6. Four spheres of radius 1 are mutually tangent. What is the radius of the smallest sphere containing them?

p7. Find the radius of the largest circle that lies above the x-axis and below the parabola $y = 2 - x^2$.

p8. For some nonzero constant a, let $f(x) = e^{ax}$ and $g(x) = \frac{1}{a}\log x$. Find all possible values of a such that the graphs of f and g are tangent at exactly one point.

p9. Consider a regular pentagon and connect each vertex to the pair of vertices farthest from it by line segments. The line segments intersect at 5 points to form another smaller pentagon. If the large pentagon has side length 1, compute the area of the smaller pentagon. Express your answer without trigonometric functions.

p10. Let f(x) be a function that satisfies $f(x)f(2 - x) = x^2 f(x - 2)$ and $f(1) = \frac{1}{403}$. Compute $f(2015)$.

p11. You are playing a game on the number line. At the beginning of the game, every real number on $[0, 4)$ is uncovered, and the rest are covered. A turn consists of picking a real number r such that, for all x where $r \le x < r + 1$, x is uncovered. The turn ends by covering all such x. At the beginning of a turn, one selects such a real r uniformly at random from among all possible choices for r,; the game ends when no such r exists. Compute the expected number of turns that will take place during this game.

p12. Consider the recurrence $a_{n+1} = 4a_n(1 - a_n)$
Call a point $a_0 \in [0, 1]$ q-*periodic* if $a_q = a_0$. For example, $a_0 = 0$ is always a q-periodic fixed point for any q. Compute the number of positive 2015-periodic fixed points.

p13. Let $a, b, c \in \{-1, 1\}$. Evaluate the following expression, where the sum is taken over all possible choices of a, b, and c:

$$\sum abc \left(2^{\frac15} + a2^{\frac25} + b2^{\frac35} + c2^{\frac45}\right)^4 .$$

p14. A small circle A of radius $\frac13$ rotates, without slipping, inside and tangent to a unit circle B. Let p be a fixed point on A, and compute the length of the closed curve traced out by p as A rotates inside B.

p15. Let x_1, x_2, x_3, x_4, x_5 be distinct positive integers such that $x_1 + x_2 + x_3 + x_4 + x_5 = 100$. Compute the maximum value of the expression

$$\frac{(x_2x_5+1)(x_3x_5+1)(x_4x_5+1)}{(x_2-x_1)(x_3-x-1)(x_4-x_1)}+\frac{(x_1x_5+1)(x_3x_5+1)(x_4x_5+1)}{(x_1-x2)(x_3-x_2)(x_4-x_2)}+\frac{(x_1x_5+1)(x_2x_5+1)(x_4x_5+1)}{(x_1-x_3)(x_2-x_3)(x_4-x_3)}+\frac{(x_1x_5}{(x_1-}$$

PS. You had better use hide for answers.

2016 **p1.** How many squares are there in the xy-plane such that both coordinates of each vertex are integers between 0 and 100 inclusive, and the sides are parallel to the axes?

p2. According to the Constitution of the Kingdom of Nepal, the shape of the flag is constructed as follows:
Draw a line AB of the required length from left to right. From A draw a line AC perpendicular to AB making AC equal to AB plus one third AB. From AC mark off D making line AD equal to line AB. Join BD. From BD mark off E making BE equal to AB. Touching E draw a line FG, starting from the point F on line AC, parallel to AB to the right hand-side. Mark off FG equal to AB. Join CG. If the length of AB is 1 unit, what is the area of the flag?

p3. You have 17 apples and 7 friends, and you want to distribute apples to your friends. The only requirement is that Steven, one of your friends, does not receive more than half of the apples. Given that apples are indistinguishable and friends are distinguishable, compute the number of ways the apples can be distributed.

p4. At $t = 0$ Tal starts walking in a line from the origin. He continues to walk forever at a rate of 1 m/s and lives happily ever after. But Michael (who is Tal's biggest fan) can't bear to say goodbye. At $t = 10$ s he starts running on Tal's path at a rate of n such that $n > 1$ m/s. Michael runs to Tal, gives him a high-five, runs back to the origin, and repeats the process forever. Assuming that the high-fives occur at time $t_0, t_1, t_2, ...,$ compute the limiting value of $\frac{t_z}{t_{z-1}}$ as $z \to \infty$.

p5. In a classroom, there are 47 students in 6 rows and 8 columns. Every student's position is expressed by (i, j). After moving, the position changes to (m, n). Define the change of every student as $(i - m) + (j - n)$. Find the maximum of the sum of changes of all students.

p6. Consider the following family of line segments on the coordinate plane. We take $(0, \pi 2 - a)$ and $(a, 0)$ to be the endpoints of any line segment in the set, for any $0 \le a \le \frac{\pi}{2}$. Let A be the union of all of these line segments. Compute the area of A.

p7. Compute the smallest $n > 2015$ such that $6^n + 8^n$ is divisible by 7.

p8. Find the radius of the largest circle that lies above the x-axis and below the parabola $y = 2 - x^2$.

p9. Let C_1 be the circle in the complex plane with radius 1 centered at 0. Let C_2 be the circle in the complex plane with radius 2 centered at $4 - 2i$. Let C_3 be the circle in the complex plane with radius 4 centered at $3 + 8i$. Let S be the set of points which are of the form $\frac{k_1+k_2+k_3}{3}$ where $k_1 \in C_1$, $k_2 \in C_2$, $k_3 \in C_3$. What is the area of S? (Note: a circle or radius r only contains the points at distance r from the center and does not include the points inside the circle)

p10. 3 points are independently chosen at random on a circle. What is the probability that they form an acute triangle?

p11. We say that a number is ascending if its digits are, from left-to-right, in nondecreasing order. We say that a number is descending if the digits are, from left-to-right, in nonincreasing order. Let an be the number of n-digit positive integers which are ascending, and bn be the number of n-digit positive integers which are descending. Compute the ordered pair (x, y) such that

$\lim_{n\to\infty} \frac{b_n}{a_n} - x_n - y = 0.$

p12. Let $f(x)$ be a function so that $f(f(x)) = \frac{2x}{1-x^2}$, and $f(x)$ is continuous at all but two points. Compute $f(\sqrt{3})$.

p13. Compute $\sum_{k=1,k\neq m}^{\infty} \frac{1}{(k+m)(k-m)}$

p14. Let $\{x\}$ denote the fractional part of x, the unique real $0 \le \{x\} < 1$ such that $x - \{x\}$ becomes integer. For the function $f_{a,b}(x) = \{x+a\} + 2\{x+b\}$, let its range be $[m_{a,b}, M_{a,b})$. Find the minimum of $M_{a,b}$ as a and b ranges along all reals.

p15. An ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is tangent to each of the circles $(x-1)^2+y^2 = 1$ and $(x+1)^2+y^2 = 1$ at two points. Find the ordered pair (a, b) that minimizes the area of the ellipse .

PS. You had better use hide for answers.