CentroAmerican 2016

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– Day1

1 Find all positive integers n that have 4 digits, all of them perfect squares, and such that n is divisible by 2, 3, 5 and 7.

2 Let ABC be an acute-angled triangle, Γ its circumcircle and M the midpoint of BC. Let N be a point in the arc BC of Γ not containing A such that $\angle NAC = \angle BAM$. Let R be the midpoint of AM, S the midpoint of AN and T the foot of the altitude through A. Prove that R, S and T are collinear.

3 The polynomial $Q(x) = x^3 - 21x + 35$ has three different real roots. Find real numbers a and b such that the polynomial $x^2 + ax + b$ cyclically permutes the roots of Q, that is, if r, s and t are the roots of Q (in some order) then $P(r) = s$, $P(s) = t$ and $P(t) = r$.

– Day 2

4 The number "3" is written on a board. Ana and Bernardo take turns, starting with Ana, to play the following game. If the number written on the board is n, the player in his/her turn must replace it by an integer m coprime with n and such that $n < m < n^2$. The first player that reaches a number greater or equal than 2016 loses. Determine which of the players has a winning strategy and describe it.

5 We say a number is irie if it can be written in the form $1 + \dfrac{1}{k}$ for some positive integer k. Prove that every integer $n \geq 2$ can be written as the product of r distinct irie numbers for every integer $r \geq n - 1$.

6 Let $\triangle ABC$ be triangle with incenter I and circumcircle Γ. Let $M = BI \cap \Gamma$ and $N = CI \cap \Gamma$, the line parallel to MN through I cuts AB, AC in P and Q. Prove that the circumradius of $\odot(BNP)$ and $\odot(CMQ)$ are equal.