Lexington Math Tournament - Team Round / Potpourri, year 2021-22

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by parmenides51, kevinmathz

– 2021 Spring Divisions

A1 Triangle LMT has $\overline{MA}$ as an altitude. Given that $MA = 16$, $MT = 20$, and $LT = 25$, find the length of the altitude from L to $\overline{MT}$.

Proposed by Kevin Zhao

A2 The function $f(x)$ has the property that $f(x) = -\frac{1}{f(x-1)}$. Given that $f(0) = -\frac{1}{21}$, find the value of $f(2021)$.

Proposed by Ada Tsui

A3 Find the greatest possible sum of integers a and b such that $\frac{2021!}{20^a \cdot 21^b}$ is a positive integer.

Proposed by Aidan Duncan

A4 B11 Five members of the Lexington Math Team are sitting around a table. Each flips a fair coin. Given that the probability that three consecutive members flip heads is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Proposed by Alex Li

A5 In rectangle $ABCD$, points E and F are on sides $\overline{BC}$ and $\overline{AD}$, respectively. Two congruent semicircles are drawn with centers E and F such that they both lie entirely on or inside the rectangle, the semicircle with center E passes through C, and the semicircle with center F passes through A. Given that $AB = 8$, $CE = 5$, and the semicircles are tangent, find the length BC.

Proposed by Ada Tsui

A6 B12 Given that the expected amount of 1s in a randomly selected 2021-digit number is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Proposed by Hannah Shen

A7 B15 A geometric sequence consists of 11 terms. The arithmetic mean of the first 6 terms is 63, and the arithmetic mean of the last 6 terms is 2016. Find the 7th term in the sequence.

Proposed by Powell Zhang

A8 Isosceles $\triangle ABC$ has interior point O such that $AO = \sqrt{52}$, $BO = 3$, and $CO = 5$. Given that $\angle ABC = 120^\circ$, find the length AB.

Proposed by Powell Zhang

A9 Find the sum of all positive integers n such that $7 < n < 100$ and 1573_n has 6 factors when written in base 10.

Proposed by Aidan Duncan

A10 Pieck the Frog hops on Pascal's Triangle, where she starts at the number 1 at the top. In a hop, Pieck can hop to one of the two numbers directly below the number she is currently on with equal probability. Given that the expected value of the number she is on after 7 hops is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Proposed by Steven Yu

A11 B17 In $\triangle ABC$ with $\angle BAC = 60^\circ$ and circumcircle ω, the angle bisector of $\angle BAC$ intersects side $\overline{BC}$ at point D, and line AD is extended past D to a point A'. Let points E and F be the feet of the perpendiculars of A' onto lines AB and AC, respectively. Suppose that ω is tangent to line EF at a point P between E and F such that $\frac{EP}{FP} = \frac{1}{2}$. Given that $EF = 6$, the area of $\triangle ABC$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m+n+p$.

Proposed by Taiki Aiba

A12 B18 There are 23 balls on a table, all of which are either red or blue, such that the probability that there are n red balls and $23-n$ blue balls on the table ($1 \le n \le 22$) is proportional to n. (e.g. the probability that there are 2 red balls and 21 blue balls is twice the probability that there are 1 red ball and 22 blue balls.) Given that the probability that the red balls and blue balls can be arranged in a line such that there is a blue ball on each end, no two red balls are next to each other, and an equal number of blue balls can be placed between each pair of adjacent red balls is $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a+b$. Note: There can be any nonzero number of consecutive blue balls at the ends of the line.

Proposed by Ada Tsui

A13 In a round-robin tournament, where any two players play each other exactly once, the fact holds that among every three students A, B, and C, one of the students beats the other two. Given that there are six players in the tournament and Aidan beats Zach but loses to Andrew, find how many ways there are for the tournament to play out. Note: The order in which the matches take place does not matter.

Proposed by Kevin Zhao

A14 Alex, Bob, and Chris are driving cars down a road at distinct constant rates. All people are driving a positive integer number of miles per hour. All of their cars are 15 feet long. It takes Alex 1 second longer to completely pass Chris than it takes Bob to completely pass Chris. The passing time is defined as the time where their cars overlap. Find the smallest possible sum of their speeds, in miles per hour.

Proposed by Sammy Charney

A15 B20 Andy and Eddie play a game in which they continuously flip a fair coin. They stop flipping when either they flip tails, heads, and tails consecutively in that order, or they flip three tails in a row. Then, if there has been an odd number of flips, Andy wins, and otherwise Eddie wins. Given that the probability that Andy wins is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Proposed by Anderw Zhao and Zachary Perry

A16 Find the number of ordered pairs (a,b) of positive integers less than or equal to 20 such that

$$\gcd(a,b) > 1 \quad \text{and} \quad \frac{1}{\gcd(a,b)} + \frac{a+b}{\text{lcm}(a,b)} \geq 1.$$

Proposed by Zachary Perry

A17 Given that the value of

$$\sum_{k=1}^{2021} \frac{1}{1^2+2^2+3^2+\cdots+k^2} + \sum_{k=1}^{1010} \frac{6}{2k^2-k} + \sum_{k=1011}^{2021} \frac{24}{2k+1}$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Proposed by Aidan Duncan

A18 Points X and Y are on a parabola of the form $y = \frac{x^2}{a^2}$ and A is the point $(x,y) = (0,a)$. Assume XY passes through A and hits the line $y = -a$ at a point B. Let ω be the circle passing through $(0,-a)$, A, and B. A point P is chosen on ω such that $PA = 8$. Given that X is between A and B, $AX = 2$, and $XB = 10$, find $PX \cdot PY$.

Proposed by Kevin Zhao

A19 Let S be the sum of all possible values of $a \cdot c$ such that

$$a^3 + 3ab^2 - 72ab + 432a = 4c^3$$

if a, b, and c are positive integers, $a+b > 11$, $a > b-13$, and $c \leq 1000$. Find the sum of all distinct prime factors of S.

Proposed by Kevin Zhao

A20 Let Ω be a circle with center O. Let ω_1 and ω_2 be circles with centers O_1 and O_2, respectively, internally tangent to Ω at points A and B, respectively, such that O_1 is on $\overline{OA}$, and O_2 is on $\overline{OB}$ and ω_1. There exists a point P on line AB such that P is on both ω_1 and ω_2. Let the external tangent of ω_1 and ω_2 on the same side of line AB as O hit ω_1 at X and ω_2 at Y, and let lines AX and BY intersect at N. Given that $O_1X = 81$ and $O_2Y = 18$, the value of $NX \cdot NA$ can be written as $a\sqrt{b} + c$, where a, b, and c are positive integers, and b is not divisible by the square of a prime. Find $a + b + c$.

Proposed by Kevin Zhao

A21 B22 A Haiku is a Japanese poem of seventeen syllables, in three lines of five, seven, and five.

In how many ways
Can you add three integers
Summing seventeen?
Order matters here.
For example, eight, three, six
Is not eight, six, three.
All nonnegative,
Do not need to be distinct.
What is your answer?

Proposed by Derek Gao

A22 B23 A Haiku is a Japanese poem of seventeen syllables, in three lines of five, seven, and five.

Ada has been told
To write down five haikus plus
Two more every hour.
Such that she needs to
Write down five in the first hour
Seven, nine, so on.
Ada has so far
Forty haikus and writes down
Seven every hour.
At which hour after
She begins will she not have
Enough haikus done?

Proposed by Ada Tsui

A23 B24 A Haiku is a Japanese poem of seventeen syllables, in three lines of five, seven, and five.

A group of haikus
Some have one syllable less

Sixteen in total.
The group of haikus
Some have one syllable more
Eighteen in total.
What is the largest
Total count of syllables
That the group can't have?
(For instance, a group
Sixteen, seventeen, eighteen
Fifty-one total.)
(Also, you can have
No sixteen, no eighteen
Syllable haikus)

Proposed by Jeff Lin

A 24 A Haiku is a Japanese poem of seventeen syllables, in three lines of five, seven, and five.

Using the four words
"Hi", "hey", "hello", and "haiku",
How many haikus
Can somebody make?
(Repetition is allowed,
Order does matter.)

Proposed by Jeff Lin

A25 B26 Chandler the Octopus is making a concoction to create the perfect ink. He adds 1.2 grams of melanin, 4.2 grams of enzymes, and 6.6 grams of polysaccharides. But Chandler accidentally added n grams of an extra ingredient to the concoction, Chemical X, to create glue. Given that Chemical X contains none of the three aforementioned ingredients, and the percentages of melanin, enzymes, and polysaccharides in the final concoction are all integers, find the sum of all possible positive integer values of n.

Proposed by Taiki Aiba

A26 B27 Chandler the Octopus along with his friends Maisy the Bear and Jeff the Frog are solving LMT problems. It takes Maisy 3 minutes to solve a problem, Chandler 4 minutes to solve a problem and Jeff 5 minutes to solve a problem. They start at $12:00$ pm, and Chandler has a dentist appointment from $12:10$ pm to $12:30$, after which he comes back and continues solving LMT problems. The time it will take for them to finish solving 50 LMT problems, in hours, is m/n ,where m and n are relatively prime positive integers. Find $m+n$.

Note: they may collaborate on problems.

Proposed by Aditya Rao

A27 Chandler the Octopus is at a tentacle party!
At this party, there is 1 creature with 2 tentacles, 2 creatures with 3 tentacles, 3 creatures with 4 tentacles, all the way up to 14 creatures with 15 tentacles. Each tentacle is distinguishable from all other tentacles. For some $2 \leq m < n \leq 15$, a creature with m tentacles "meets" a creature with n tentacles; "meeting" another creature consists of shaking exactly 1 tentacle with each other. Find the number of ways there are to pick distinct $m < n$ between 2 and 15, inclusive, and then to pick a creature with m tentacles to "meet" a selected creature with n tentacles.

Proposed by Armaan Tipirneni, Richard Chen, and Denise the Octopus

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A28 B29 Addison and Emerson are playing a card game with three rounds. Addison has the cards $1, 3$, and 5, and Emerson
has the cards $2, 4$, and 6. In advance of the game, both designate each one of their cards to be played for either round one, two, or three. Cards cannot be played for multiple rounds. In each round, both show each other their designated card for that round, and the person with the higher-numbered card wins the round. The person who wins the most rounds wins the game. Let m/n be the probability that Emerson wins, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by Ada Tsui

A29 B30 In a group of 6 people playing the card game Tractor, all 54 cards from 3 decks are dealt evenly to all the players
at random. Each deck is dealt individually. Let the probability that no one has at least two of the same card be X.
Find the largest integer n such that the nth root of X is rational.

Proposed by Sammy Charney

Due to the problem having infinitely many solutions, all teams who inputted answers received points.

A30 Ryan Murphy is playing poker. He is dealt a hand of 5 cards. Given that the probability that he has a straight hand (the ranks are all consecutive; e.g. $3, 4, 5, 6, 7$ or $9, 10, J, Q, K$) or 3 of a kind (at least 3 cards of the same rank; e.g. $5, 5, 5, 7, 7$ or $5, 5, 5, 7, K$) is m/n , where m and n are relatively prime positive integers, find $m + n$.

Proposed by Aditya Rao

B1 Given that the expression $\frac{20^{21}}{20^{20}} + \frac{20^{20}}{20^{21}}$ can be written in the form m/n , where m and n are relatively prime positive integers, find $m + n$.

Proposed by Ada Tsui

B2 Find the greatest possible distance between any two points inside or along the perimeter of an equilateral triangle with side length 2.

Proposed by Alex Li

B3 Aidan rolls a pair of fair, six sided dice. Let$n$ be the probability that the product of the two numbers at the top is prime. Given that n can be written as a/b , where a and b are relatively prime positive integers, find $a+b$.

Proposed by Aidan Duncan

B4 Set S contains exactly 36 elements in the form of $2^m \cdot 5^n$ for integers $0 \le m,n \le 5$. Two distinct elements of S are randomly chosen. Given that the probability that their product is divisible by 10^7 is a/b, where a and b are relatively prime positive integers, find $a+b$.

Proposed by Ada Tsui

B5 Find the number of ways there are to permute the elements of the set $\{1,2,3,4,5,6,7,8,9\}$ such that no two adjacent numbers are both even or both odd.

Proposed by Ephram Chun

B6 Maisy is at the origin of the coordinate plane. On her first step, she moves 1 unit up. On her second step, she moves 1 unit to the right. On her third step, she moves 2 units up. On her fourth step, she moves 2 units to the right. She repeats this pattern with each odd-numbered step being 1 unit more than the previous step. Given that the point that Maisy lands on after her 21st step can be written in the form (x,y), find the value of $x+y$.

Proposed by Audrey Chun

B7 Given that x and y are positive real numbers such that $\frac{5}{x} = \frac{y}{13} = \frac{x}{y}$, find the value of x^3+y^3.

Proposed by Ephram Chun

B8 Find the number of arithmetic sequences a_1,a_2,a_3 of three nonzero integers such that the sum of the terms in the sequence is equal to the product of the terms in the sequence.

Proposed by Sammy Charney

B9 Convex pentagon $PQRST$ has $PQ = TP = 5$, $QR = RS = ST = 6$, and $\angle QRS = \angle RST = 90^o$. Given that points U and V exist such that $RU = UV = VS = 2$, find the area of pentagon $PQUVT$.

Proposed by Kira Tang

B10 Let $f(x)$ be a function mapping real numbers to real numbers. Given that $f(f(x)) = \frac{1}{3x}$, and $f(2) = \frac{1}{9}$, find $f\left(\frac{1}{6}\right)$.

Proposed by Zachary Perry

B13 Call a 4-digit number $\overline{abcd}$ *unnoticeable* if $a+c=b+d$ and $\overline{abcd}+\overline{cdab}$ is a multiple of 7. Find the number of unnoticeable numbers.

Note: a, b, c, and d are nonzero distinct digits.

Proposed by Aditya Rao

B14 In the expansion of $(2x+3y)^{20}$, find the number of coefficients divisible by 144.

Proposed by Hannah Shen

B16 Bob plants two saplings. Each day, each sapling has a $1/3$ chance of instantly turning into a tree. Given that the expected number of days it takes both trees to grow is m/n , where m and n are relatively prime positive integers, find $m+n$.

Proposed by Powell Zhang

B19 Kevin is at the point $(19,12)$. He wants to walk to a point on the ellipse $9x^2+25y^2 = 8100$, and then walk to $(-24,0)$. Find the shortest length that he has to walk.

Proposed by Kevin Zhao

B21 A Haiku is a Japanese poem of seventeen syllables, in three lines of five, seven, and five.

Take five good haikus
Scramble their lines randomly
What are the chances
That you end up with
Five completely good haikus
(With five, seven, five)?
Your answer will be
m over n where m,n
Are numbers such that
m,n positive
Integers where gcd
Of m,n is 1.
Take this answer and
Add the numerator and
Denominator.

Proposed by Jeff Lin

B28 Maisy and Jeff are playing a game with a deck of cards with 4 0's, 4 1's, 4 2's, all the way up to 4 9's. You cannot tell apart cards of the same number. After shuffling the deck, Maisy and Jeff each take 4 cards, make the largest 4-digit integer they can, and then compare. The person with the larger 4-digit integer wins. Jeff goes first and draws the cards $2, 0, 2, 1$ from the deck. Find the number of hands Maisy can draw to beat that, if the order in which she draws the cards matters.

Proposed by Richard Chen

– 2021 Fall

1 Kevin writes the multiples of three from 1 to 100 on the whiteboard. How many digits does he write?

2 How many ways are there to permute the letters $\{S, C, R, A, M, B, L, E\}$ without the permutation containing the substring $LAME$?

3 Farmer Boso has a busy farm with lots of animals. He tends to $5b$ cows, $5a + 7$ chickens, and b^{a-5} insects. Note that each insect has 6 legs. The number of cows is equal to the number of insects. The total number of legs present amongst his animals can be expressed as $\overline{LLL} + 1$, where L stands for a digit. Find L.

4 Segment AB of length 13 is the diameter of a semicircle. Points C and D are located on the semicircle but not on segment AB. Segments AC and BD both have length 5. Given that the length of CD can be expressed as $\frac{a}{b}$ where a and b are relatively prime positive integers, find $a + b$.

5 In rectangle $ABCD$, $AB = 40$ and $AD = 30$. Let C' be the reflection of C over BD. Find the length of AC'.

6 Call a polynomial $p(x)$ with positive integer roots *corrupt* if there exists an integer that cannot be expressed as a sum of (not necessarily positive) multiples of its roots. The polynomial $A(x)$ is monic, corrupt, and has distinct roots. As well, $A(0)$ has 7 positive divisors. Find the least possible value of $|A(1)|$.

7 Let $n = 6901$. There are 6732 positive integers less than or equal to n that are also relatively prime to n. Find the sum of the distinct prime factors of n.

8 Three distinct positive integers are chosen at random from $\{1, 2, 3..., 12\}$. The probability that no two elements of the set have an absolute difference less than or equal to 2 can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a + b$.

9 Points X and Y on the unit circle centered at $O = (0, 0)$ are at $(-1, 0)$ and $(0, -1)$ respectively. Points P and Q are on the unit circle such that $\angle PXO = \angle QYO = 30^o$. Let Z be the intersection of line XP and line YQ. The area bounded by segment ZP, segment ZQ, and arc PQ can be expressed as $a\pi - b$ where a and b are rational numbers. Find $\frac{1}{ab}$.

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10 There are 15 people attending math team: 12 students and 3 captains. One of the captains brings 33 identical snacks. A nonnegative number of names (students and/or captains) are written on the NO SNACK LIST. At the end of math team, all students each get n snacks, and all captains get $n+1$ snacks, unless the person's name is written on the board. After everyone's snacks are distributed, there are none left. Find the number of possible integer values of n.

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11 The LHS Math Team is going to have a Secret Santa event! Nine members are going to participate, and each person must give exactly one gift to a specific recipient so that each person receives exactly one gift. But to make it less boring, no pairs of people can just swap gifts. The number of ways to assign who gives gifts to who in the Secret Santa Exchange with these constraints is N. Find the remainder when N is divided by 1000.

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12 Let x, y, and z be three not necessarily real numbers that satisfy the following system of equations: $x^3 - 4 = (2y + 1)^2$ $y^3 - 4 = (2z + 1)^2$ $z^3 - 4 = (2x + 1)^2$.
Find the greatest possible real value of $(x - 1)(y - 1)(z - 1)$.

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13 Find the sum of

$$\frac{\sigma(n) \cdot d(n)}{\phi(n)}$$

over all positive n that divide 60.

Note: The function $d(i)$ outputs the number of divisors of i, $\sigma(i)$ outputs the sum of the factors of i, and $\phi(i)$ outputs the number of positive integers less than or equal to i that are relatively prime to i.

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14 In a cone with height 3 and base radius 4, let X be a point on the circumference of the base. Let Y be a point on the surface of the cone such that the distance from Y to the vertex of the cone is 2, and Y is diametrically opposite X with respect to the base of the cone. The length of the shortest path across the surface of the cone from X to Y can be expressed as $\sqrt{a + \sqrt{b}}$, where a and b are positive integers. Find $a + b$.

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15 There are 28 students who have to be separated into two groups such that the number of students in each group
is a multiple of 4. The number of ways to split them into the groups can be written as

$$\sum_{k\geq 0} 2^k a_k = a_0 + 2a_1 + 4a_2 + ...$$

where each a_i is either 0 or 1. Find the value of

$$\sum_{k\geq 0} ka_k = 0 + a_1 + 2a_2 + 3a3_+ \ldots.$$

– 2022 Spring

1 Derek and Jacob have a cake in the shape a rectangle with dimensions 14 inches by 9 inches. They make a deal to split it: Derek takes home the portion of the cake that is less than one inch from the border, while Jacob takes home the remainder of the cake. Let $D : J$ be the ratio of the amount of cake Derek took to the amount of cake Jacob took, where D and J are relatively prime positive integers. Find $D + J$.

2 Five people are standing in a straight line, and the distance between any two people is a unique positive integer number of units. Find the least possible distance between the leftmost and rightmost people in the line in units.

3 Let the four real solutions to the equation $x^2 + \frac{144}{x^2} = 25$ be r_1, r_2, r_3, and r_4. Find $|r_1| + |r_2| + |r_3| + |r_4|$.

4 Jeff has a deck of 12 cards: 4 Ls, 4 Ms, and 4 Ts. Armaan randomly draws three cards without replacement. The probability that he takes 3 Ls can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

5 Find the sum

$$\sum_{n=1}^{2020} \gcd(n^3 - 2n^2 + 2021, n^2 - 3n + 3).$$

6 For all y, define cubic $f_y(x)$ such that $f_y(0) = y$, $f_y(1) = y + 12$, $f_y(2) = 3y^2$, $f_y(3) = 2y + 4$. For all y, $f_y(4)$ can be expressed in the form $ay^2 + by + c$ where a, b, c are integers. Find $a + b + c$.

7 Kevin has a square piece of paper with creases drawn to split the paper in half in both directions, and then each of the four small formed squares diagonal creases drawn, as shown below.
`https://cdn.artofproblemsolving.com/attachments/2/2/70d6c54e86856af3a977265a8054fd9b0444b png`
Find the sum of the corresponding numerical values of figures below that Kevin can create by folding the above piece
of paper along the creases. (The figures are to scale.) Kevin cannot cut the paper or rip it in any way.
`https://cdn.artofproblemsolving.com/attachments/a/c/e0e62a743c00d35b9e6e2f702106016b9e787 png`

8 The 53-digit number

$$37,984,318,966,591,152,105,649,545,470,741,788,308,402,068,827,142,719$$

can be expressed as n^21 where n is a positive integer. Find n.

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9 Let $r_1, r_2, ..., r_{2021}$ be the not necessarily real and not necessarily distinct roots of $x^{2022}+2021x = 2022$. Let $S_i = r_i^{2021} + 2022r_i$ for all $1 \le i \le 2021$. Find $\left|\sum_{i=1}^{2021} S_i\right| = |S_1 + S_2 + ... + S_{2021}|$.

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10 In a country with 5 distinct cities, there may or may not be a road between each pair of cities. It's possible to get from any city to any other city through a series of roads, but there is no set of three cities $\{A, B, C\}$ such that there are roads between A and B, B and C, and C and A. How many road systems between the five cities are possible?

– 2022 Fall

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1 Let x be the positive integer satisfying $5^2 + 28^2 + 39^2 = 24^2 + 35^2 + x^2$. Find x.

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2 Ada rolls a standard 4-sided die 5 times. The probability that the die lands on at most two distinct sides can be written as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A + B$

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3 Billiam is distributing his ample supply of balls among an ample supply of boxes. He distributes the balls as follows: he places a ball in the first empty box, and then for the greatest positive integer n such that all n boxes from box 1 to box n have at least one ball, he takes all of the balls in those n boxes and puts them into box $n + 1$. He then repeats this process indefinitely. Find the number of repetitions of this process it takes for one box to have at least 2022 balls.

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4 Find the least positive integer ending in 7 with exactly 12 positive divisors.

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5 Let H be a regular hexagon with side length 1. The sum of the areas of all triangles whose vertices are all vertices of H can be expressed as $A\sqrt{B}$ for positive integers A and B such that B is square-free. What is $1000A + B$?

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6 An isosceles trapezoid $PQRS$, with $\overline{PQ} = \overline{QR} = \overline{RS}$ and $\angle PQR = 120^o$, is inscribed in the graph of $y = x^2$ such that QR is parallel to the x-axis and R is in the first quadrant. The x-coordinate of point R can be written as $\frac{\sqrt{A}}{B}$ for positive integers A and B such that A is square-free. Find $1000A + B$.

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7 A regular hexagon is split into 6 congruent equilateral triangles by drawing in the 3 main diagonals. Each triangle is colored 1 of 4 distinct colors. Rotations and reflections of the figure are considered nondistinct. Find the number of possible distinct colorings.

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8 An odd positive integer n can be expressed as the sum of two or more consecutive integers in exactly 2023 ways. Find the greatest possible nonnegative integer k such that 3^k is a factor of the least possible value of n.

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9 In isosceles trapezoid $ABCD$ with $AB < CD$ and $BC = AD$, the angle bisectors of $\angle A$ and $\angle B$ intersect CD at E and F respectively, and intersect each other outside the trapezoid at G. Given that $AD = 8$, $EF = 3$, and $EG = 4$, the area of $ABCD$ can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, and c, with a and c relatively prime and b squarefree. Find $10000a + 100b + c$.

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10 Let $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$ and $\beta = \sin^{-1}\left(\frac{3}{5}\right)$.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\cos(\alpha n + \beta m)}{2^n 3^m}$$

can be written as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A + B$.

– 2023 Spring

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1 Given the following system of equations:

$$\begin{cases} RI + G + SP = 50 \\ RI + T + M = 63 \\ G + T + SP = 25 \\ SP + M = 13 \\ M + RI = 48 \\ N = 1 \end{cases}$$

Find the value of L that makes $LMT + SPRING = 2023$ true.

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2 How many integers of the form n^{2023-n} are perfect squares, where n is a positive integer between 1 and 2023 inclusive?

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3 Beter Pai wants to tell you his fastest 40-line clear time in Tetris, but since he does not want Qep to realize she is
better at Tetris than he is, he does not tell you the time directly. Instead, he gives you the following requirements,
given that the correct time is t seconds: • $t < 100$. • t is prime. • $t - 1$ has 5 proper factors. • all prime factors of $t + 1$ are single digits. • $t - 2$ is a multiple of 3. • $t + 2$ has 2 factors.
Find t.

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4 There exists a certain right triangle with the smallest area in the 2D coordinate plane such that all of its vertices have integer coordinates but none of its sides are parallel to the x- or y-axis. Additionally, all of its sides have distinct, integer lengths. What is the area of this triangle?

5 How many ways are there to place the integers from 1 to 8 on the vertices of a regular octagon such that the sum of the numbers on any 4 vertices forming a rectangle is even? Rotations and reflections of the same arrangement are considered distinct

6 Find the least positive integer m such that $105|9^{(p^2)} - 29^p + m$ for all prime numbers $p > 3$.

7 Jerry writes down all binary strings of length 10 without any two consecutive 1s. How many 1s does Jerry write?

8 Let x, y, and z be positive reals that satisfy the system

$$\begin{cases} x^2 + xy + y^2 = 10 \\ x^2 + xz + z^2 = 20 \\ y^2 + yz + z^2 = 30 \end{cases}$$

Find $xy + yz + xz$.

9 In $\triangle ABC$, $AB = 13$, $BC = 14$, and $CA = 15$. Let E and F be the feet of the altitudes from B onto CA, and C onto AB, respectively. A line ℓ is parallel to EF and tangent to the circumcircle of ABC on minor arc BC. Let X and Y be the intersections of ℓ with AB and AC respectively. Find XY .

10 The sequence $a_0, a_1, a_2, ...$ is defined such that $a_0 = 2 + \sqrt{3}$, $a_1 = \sqrt{5 - 2\sqrt{5}}$, and

$$a_n a_{n-1} a_{n-2} - a_n + a_{n-1} + a_{n-2} = 0.$$

Find the least positive integer n such that $a_n = 1$.

– 2023 Fall

1 George has 150 cups of flour and 200 eggs. He can make a cupcake with 3 cups of flour and 2 eggs, or he can make an omelet with 4 eggs. What is the maximum number of treats (both omelets and cupcakes) he canmake?

2 For how many nonnegative integer values of k does the equation $7x^2 + kx + 11 = 0$ have no real solutions?

3 Adamand Topher are playing a game in which each of them starts with 2 pickles. Each turn, they flip a fair coin: if it lands heads, Topher takes 1 pickle from Adam; if it lands tails, Adam takes 2 pickles from Topher. (If Topher has only 1 pickle left, Adam will just take it.) What's the probability that Topher will have all 4 pickles before Adam does?

4 Fred chooses a positive two-digit number with distinct nonzero digits. Laura takes Fred's number and swaps its digits. She notices that the sum of her number and Fred's number is a perfect square and the positive difference between them is a perfect cube. Find the greater of the two numbers.

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5 In regular hexagon $ABCDEF$ with side length 2, let P, Q, R, and S be the feet of the altitudes from A to BC, EF, CF, and BE, respectively. Find the area of quadrilateral $PQRS$.

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6 Jeff rolls a standard 6 sided die repeatedly until he rolls either all of the prime numbers possible at least once, or all the of even numbers possible at least once. Find the probability that his last roll is a 2.

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7 How many 2-digit factors does 555555 have?

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8 Let J , E, R, and Y be four positive integers chosen independently and uniformly at random from the set of factors of 1428. What is the probability that $JERRY = 1428$? Express your answer in the form $\frac{a}{b\cdot 2^n}$ where n is a nonnegative integer, $a$and b are odd, and gcd $(a,b) = 1$.

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9 In triangle ABC, let O be the circumcenter and let G be the centroid. The line perpendicular to OG at O intersects BC at M such that M, G, and A are collinear and $OM = 3$. Compute the area of ABC, given that $OG = 1$.

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10 Aidan and Andrew independently select distinct cells in a 4 by 4 grid, as well as a direction (either up, down, left, or right), both at random. Every second, each of them will travel 1 cell in their chosen direction. Find the probability that Aidan and Andrew willmeet (be in the same cell at the same time) before either one of them hits an edge of the grid. (If Aidan and Andrew cross paths by switching cells, it doesn't count as meeting.)

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11 Find the number of degree 8 polynomials $f(x)$ with nonnegative integer coefficients satisfying both $f(1) = 16$ and $f(-1) = 8$.

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12 In triangle ABC with $AB = 7$, $AC = 8$, and $BC = 9$, the A-excircle is tangent to BC at point D and also tangent to lines AB and AC at points and F, respectively. Find $[DEF]$. (The A-excircle is the circle tangent to segment BC and the extensions of rays AB and AC. Also, $[XYZ]$ denotes the area of triangle XYZ.)

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13 Ella lays out 16 coins heads up in a 4×4 grid as shown.
https://cdn.artofproblemsolving.com/attachments/3/3/a728be9c51b27f442109cc8613ef50d61182a
png
On a move, Ella can flip all the coins in any row, column, or diagonal (including small diagonals such as H_1 & H_4). If rotations are considered distinct, how many distinct grids of coins can she create in a finite number of moves?

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14 Find

$$\sum_{i=1}^{100} i \gcd(i, 100).$$

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15 In triangle ABC with $AB = 26$, $BC = 28$, and $CA = 30$, let M be the midpoint of AB and let N be the midpoint of CA. The circumcircle of triangle BCM intersects AC at $X \neq C$, and the circumcircle of triangle BCNintersects AB at $Y \neq B$. Lines MX and NY intersect BC at P and Q, respectively. The area of quadrilateral $PQYX$ can be expressed as $\frac{p}{q}$ for positive integers p and q such that gcd$(p, q) = 1$. Find q.
