Math Open At Andover 2021

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by parmenides51, andyxpandy99

– Speed Round

1 What is $2021 + 20 + 21 + 2 + 0 + 2 + 1$?

Proposed by Nathan Xiong

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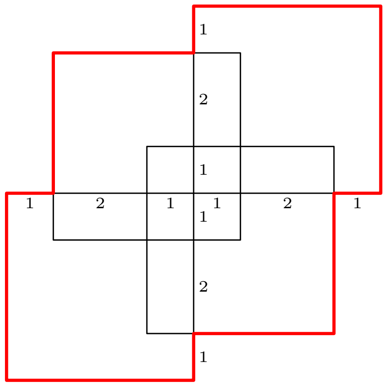

Compute the area of the resulting shape, drawn in red above.

Proposed by Nathan Xiong

3 Find the number of ordered pairs (x, y), where x and y are both integers between 1 and 9, inclusive, such that the product $x \times y$ ends in the digit 5.

Proposed by Andrew Wen

4 Let a, b, and c be real numbers such that $0 \le a, b, c \le 5$ and $2a + b + c = 10$. Over all possible values of a, b, and c, determine the maximum possible value of $a + 2b + 3c$.

Proposed by Andrew Wen

5 There are 12 students in Mr. DoBa's math class. On the final exam, the average score of the top 3 students was 8 more than the average score of the other students, and the average score of the entire class was 85. Compute the average score of the top 3 students.

Proposed by Yifan Kang

6 Suppose (a, b) is an ordered pair of integers such that the three numbers a, b, and ab form an arithmetic progression, in that order. Find the sum of all possible values of a.

Proposed by Nathan Xiong

7 If positive real numbers x, y, z satisfy the following system of equations, compute $x + y + z$.

$$xy + yz = 30$$
$$yz + zx = 36$$
$$zx + xy = 42$$

Proposed by Nathan Xiong

8 Andrew chooses three (not necessarily distinct) integers a, b, and c independently and uniformly at random from $\{1, 2, 3, 4, 5, 6, 7\}$. Let p be the probability that $abc(a + b + c)$ is divisible by 4. If p can be written as $\frac{m}{n}$ for relatively prime positive integers m and n, then compute $m + n$.

Proposed by Andrew Wen

9 Triangle $\triangle ABC$ has $\angle A = 90^\circ$ with $BC = 12$. Square $BCDE$ is drawn such that A is in its interior. The line through A tangent to the circumcircle of $\triangle ABC$ intersects CD and BE at P and Q, respectively. If $PA = 4 \cdot QA$, and the area of $\triangle ABC$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, then compute $m + n$.

Proposed by Andy Xu

10 Let $ABCD$ be a unit square in the plane. Points X and Y are chosen independently and uniformly at random on the perimeter of $ABCD$. If the expected value of the area of triangle $\triangle AXY$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, compute $m + n$.

Proposed by Nathan Xiong

– Accuracy Round

1 Evaluate

$$2 \times (2 \times (2 \times (2 \times (2 \times (2 \times 2 - 2) - 2) - 2) - 2) - 2) - 2.$$

Proposed by Nathan Xiong

2 On Andover's campus, Graves Hall is 60 meters west of George Washington Hall, and George Washington Hall is 80 meters north of Paresky Commons. Jessica wants to walk from Graves Hall to Paresky Commons. If she first walks straight from Graves Hall to George Washington

Hall and then walks straight from George Washington Hall to Paresky Commons, it takes her 8 minutes and 45 seconds while walking at a constant speed. If she walks with the same speed directly from Graves Hall to Paresky Commons, how much time does she save, in seconds?

Proposed by Nathan Xiong

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3 Arnav is placing three rectangles into a 3×3 grid of unit squares. He has a 1×3 rectangle, a 1×2 rectangle, and a 1×1 rectangle. He must place the rectangles onto the grid such that the edges of the rectangles align with the gridlines of the grid. If he is allowed to rotate the rectangles, how many ways can he place the three rectangles into the grid, without overlap?

Proposed by William Yue

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4 Compute the number of two-digit numbers $\overline{ab}$ with nonzero digits a and b such that a and b are both factors of $\overline{ab}$.

Proposed by Nathan Xiong

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5 If x, y, z are nonnegative integers satisfying the equation below, then compute $x+y+z$.

$$\left(\frac{16}{3}\right)^x \times \left(\frac{27}{25}\right)^y \times \left(\frac{5}{4}\right)^z = 256.$$

Proposed by Jeffrey Shi

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6 Let $\triangle ABC$ be a triangle in a plane such that $AB = 13$, $BC = 14$, and $CA = 15$. Let D be a point in three-dimensional space such that $\angle BDC = \angle CDA = \angle ADB = 90^\circ$. Let d be the distance from D to the plane containing $\triangle ABC$. The value d^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Proposed by William Yue

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7 Jeffrey rolls fair three six-sided dice and records their results. The probability that the mean of these three numbers is greater than the median of these three numbers can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Proposed by Nathan Xiong

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8 Will has a magic coin that can remember previous flips. If the coin has already turned up heads m times and tails n times, the probability that the next flip turns up heads is exactly $\frac{m+1}{m+n+2}$. Suppose that the coin starts at 0 flips. The probability that after 10 coin flips, heads and tails have both turned up exactly 5 times can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Proposed by Nathan Xiong

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9 Let S be the set of ordered pairs (x, y) of positive integers for which $x + y \leq 20$. Evaluate

$$\sum_{(x,y)\in S} (-1)^{x+y} xy.$$

Proposed by Andrew Wen

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10 In $\triangle ABC$, let X and Y be points on segment BC such that $AX = XB = 20$ and $AY = YC = 21$. Let J be the A-excenter of triangle $\triangle AXY$. Given that J lies on the circumcircle of $\triangle ABC$, the length of BC can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by Andrew Wen

– Team Round

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1 The value of

$$\frac{1}{20} - \frac{1}{21} + \frac{1}{20 \times 21}$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by Nathan Xiong

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2 Four students Alice, Bob, Charlie, and Diana want to arrange themselves in a line such that Alice is at either end of the line, i.e., she is not in between two students. In how many ways can the students do this?

Proposed by Nathan Xiong

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3 For two real numbers x and y, let $x \circ y = \frac{xy}{x+y}$. The value of

$$1 \circ (2 \circ (3 \circ (4 \circ 5)))$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by Nathan Xiong

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4 Compute the number of ordered triples (x, y, z) of integers satisfying

$$x^2 + y^2 + z^2 = 9.$$

Proposed by Nathan Xiong

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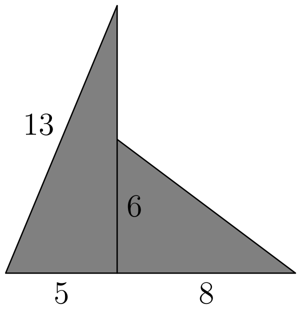
5 Two right triangles are placed next to each other to form a quadrilateral as shown. What is the perimeter of the quadrilateral?



Proposed by Nathan Xiong

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6 Find the sum of all two-digit prime numbers whose digits are also both prime numbers.

Proposed by Nathan Xiong

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7 Compute the number of ordered pairs (a, b) of positive integers satisfying $a^b = 2^{100}$.

Proposed by Nathan Xiong

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8 Evaluate

$$2^7 \times 3^0 + 2^6 \times 3^1 + 2^5 \times 3^2 + \cdots + 2^0 \times 3^7.$$

Proposed by Nathan Xiong

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9 Mr. DoBa has a bag of markers. There are 2 blue, 3 red, 4 green, and 5 yellow markers. Mr. DoBa randomly takes out two markers from the bag. The probability that these two markers are different colors can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by Raina Yang

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10 For how many nonempty subsets $S \subseteq \{1, 2, \ldots, 10\}$ is the sum of all elements in S even?

Proposed by Andrew Wen

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11 Find the product of all possible real values for k such that the system of equations

$$x^2 + y^2 = 80$$

$$x^2 + y^2 = k + 2x - 8y$$

has exactly one real solution (x, y).

Proposed by Nathan Xiong

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12 Let $\triangle ABC$ have $AB = 9$ and $AC = 10$. A semicircle is inscribed in $\triangle ABC$ with its center on segment BC such that it is tangent AB at point D and AC at point E. If $AD = 2DB$ and r is the radius of the semicircle, r^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by Andy Xu

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13 Bob has 30 identical unit cubes. He can join two cubes together by gluing a face on one cube to a face on the other cube. He must join all the cubes together into one connected solid. Over all possible solids that Bob can build, what is the largest possible surface area of the solid?

Proposed by Nathan Xiong

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14 Evaluate

$$\left\lfloor \frac{1\times 5}{7} \right\rfloor + \left\lfloor \frac{2\times 5}{7} \right\rfloor + \left\lfloor \frac{3\times 5}{7} \right\rfloor + \cdots + \left\lfloor \frac{100\times 5}{7} \right\rfloor.$$

Proposed by Nathan Xiong

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15 Consider the polynomial

$$P(x) = x^3 + 3x^2 + 6x + 10.$$

Let its three roots be a, b, c. Define $Q(x)$ to be the monic cubic polynomial with roots ab, bc, ca. Compute $|Q(1)|$.

Proposed by Nathan Xiong

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16 Let $\triangle ABC$ have $\angle ABC = 67^\circ$. Point X is chosen such that $AB = XC$, $\angle XAC = 32^\circ$, and $\angle XCA = 35^\circ$. Compute $\angle BAC$ in degrees.

Proposed by Raina Yang

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17 Compute the remainder when 10^{2021} is divided by 10101.

Proposed by Nathan Xiong

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18 Let $\triangle ABC$ be a triangle with side length $BC = 4\sqrt{6}$. Denote ω as the circumcircle of $\triangle ABC$. Point D lies on ω such that AD is the diameter of ω. Let N be the midpoint of arc BC that contains A. H is the intersection of the altitudes in $\triangle ABC$ and it is given that $HN = HD = 6$. If the area of $\triangle ABC$ can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers with a and c relatively prime and b not divisible by the square of any prime, compute $a + b + c$.

Proposed by Andy Xu

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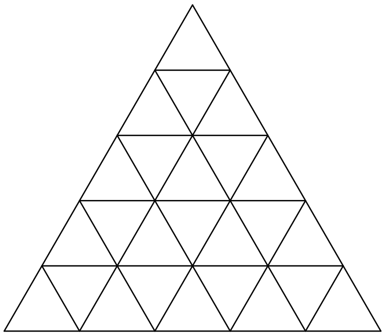
19 Consider the 5 by 5 by 5 equilateral triangular grid as shown:



Ethan chooses two distinct upward-oriented equilateral triangles bounded by the gridlines. The probability that Ethan chooses two triangles that share exactly one vertex can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Proposed by Andrew Wen

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20 Compute the sum of all integers x for which there exists an integer y such that

$$x^3+xy+y^3=503.$$

Proposed by Nathan Xiong

– Gunga Bowl Round

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1 Evaluate $2\times 0+2\times 1+2+0\times 2+1$.

Proposed by Nathan Xiong

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2 Add one pair of brackets to the expression

$$1+2\times 3+4\times 5+6$$

so that the resulting expression has a valid mathematical value, e.g., $1+2\times(3+4\times 5)+6=53$. What is the largest possible value that one can make?

Proposed by Nathan Xiong

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3 What is the last digit of 2021^{2021}?

Proposed by Yifan Kang

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4 How many of the following capital English letters look the same when rotated 180° about their center?

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Proposed by William Yue

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5 Joshua rolls two dice and records the product of the numbers face up. The probability that this product is composite can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Proposed by Nathan Xiong

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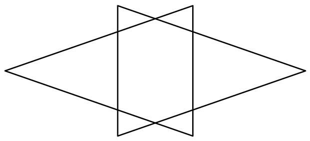
6 Determine the number of triangles, of any size and shape, in the following figure:



Proposed by William Yue

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7 Andover has a special weather forecast this week. On Monday, there is a $\frac{1}{2}$ chance of rain. On Tuesday, there is a $\frac{1}{3}$ chance of rain. This pattern continues all the way to Sunday, when there is a $\frac{1}{8}$ chance of rain. The probability that it doesn't rain in Andover all week can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Proposed by Nathan Xiong

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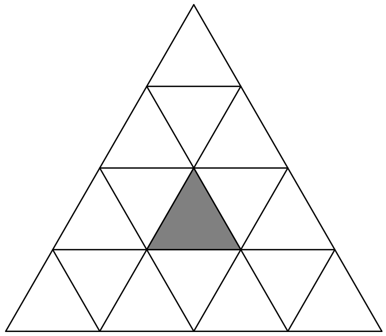
8 Compute the number of triangles of different sizes which contain the gray triangle in the figure below.



Proposed by Nathan Xiong

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9 William is biking from his home to his school and back, using the same route. When he travels to school, there is an initial 20° incline for 0.5 kilometers, a flat area for 2 kilometers, and a 20° decline for 1 kilometer. If William travels at 8 kilometers per hour during uphill 20° sections, 16 kilometers per hours during flat sections, and 20 kilometers per hour during downhill 20° sections, find the closest integer to the number of minutes it take William to get to school and back.

Proposed by William Yue

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10 We say that an ordered pair (a, b) of positive integers with $a > b$ is square-ish if both $a+b$ and $a-b$ are perfect squares. For example, $(17, 8)$ is square-ish because $17+8=25$ and $17-8=9$ are both perfect squares. How many square-ish pairs (a, b) with $a+b<100$ are there?

Proposed by Nathan Xiong

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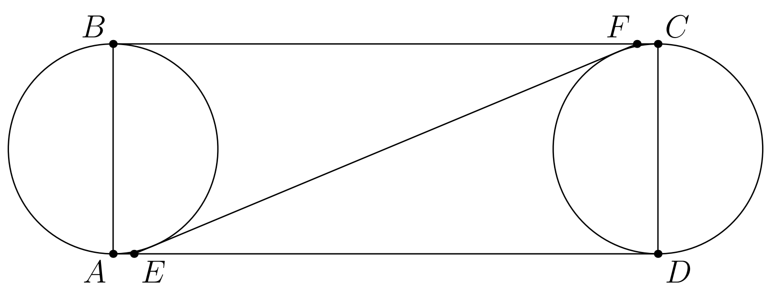
11 Let $ABCD$ be a rectangle with $AB=10$ and $BC=26$. Let ω_1 be the circle with diameter $\overline{AB}$ and ω_2 be the circle with diameter $\overline{CD}$. Suppose ℓ is a common internal tangent to ω_1 and ω_2 and that ℓ intersects AD and BC at E and F respectively. What is EF?



Proposed by Nathan Xiong

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12 Andy wishes to open an electronic lock with a keypad containing all digits from 0 to 9. He knows that the password registered in the system is 2469. Unfortunately, he is also aware that exactly two different buttons (but he does not know which ones) $\underline{a}$ and $\underline{b}$ on the keypad are broken – when $\underline{a}$ is pressed the digit b is registered in the system, and when $\underline{b}$ is pressed the digit a is registered in the system. Find the least number of attempts Andy needs to surely be able to open the lock.

Proposed by Andrew Wen

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13 Determine the greatest power of 2 that is a factor of $3^{15}+3^{11}+3^{6}+1$.

Proposed by Nathan Xiong

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14 Sinclair starts with the number 1. Every minute, he either squares his number or adds 1 to his number, both with equal probability. What is the expected number of minutes until his number is divisible by 3?

Proposed by Nathan Xiong

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15 Let a, b, c, d be the four roots of the polynomial

$$x^4 + 3x^3 - x^2 + x - 2.$$

Given that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{2}$ and $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} = -\frac{3}{4}$, the value of

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3}$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by Nathan Xiong

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16 Let $1, 7, 19, \ldots$ be the sequence of numbers such that for all integers $n \geq 1$, the average of the first n terms is equal to the nth perfect square. Compute the last three digits of the 2021st term in the sequence.

Proposed by Nathan Xiong

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17 Isosceles trapezoid $ABCD$ has side lengths $AB = 6$ and $CD = 12$, while $AD = BC$. It is given that O, the circumcenter of $ABCD$, lies in the interior of the trapezoid. The extensions of lines AD and BC intersect at T. Given that $OT = 18$, the area of $ABCD$ can be expressed as $a + b\sqrt{c}$ where a, b, and c are positive integers where c is not divisible by the square of any prime. Compute $a + b + c$.

Proposed by Andrew Wen

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18 Find the largest positive integer n such that the number $(2n)!$ ends with 10 more zeroes than the number $n!$.

Proposed by Andy Xu

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19 Let S be the set of triples (a, b, c) of non-negative integers with $a+b+c$ even. The value of the sum

$$\sum_{(a,b,c)\in S} \frac{1}{2^a 3^b 5^c}$$

can be expressed as $\frac{m}{n}$ for relative prime positive integers m and n. Compute $m + n$.

Proposed by Nathan Xiong

20 In the interior of square $ABCD$ with side length 1, a point P is chosen such that the lines ℓ_1, ℓ_2 through P parallel to AC and BD, respectively, divide the square into four distinct regions, the smallest of which has area $\mathcal{R}$. The area of the region of all points P for which $\mathcal{R} \geq \frac{1}{6}$ can be expressed as $\frac{a-b\sqrt{c}}{d}$ where $\gcd(a, b, d) = 1$ and c is not divisible by the square of any prime. Compute $a + b + c + d$.

Proposed by Andrew Wen

21 King William is located at $(1, 1)$ on the coordinate plane. Every day, he chooses one of the eight lattice points closest to him and moves to one of them with equal probability. When he exits the region bounded by the x, y axes and $x+y = 4$, he stops moving and remains there forever. Given that after an arbitrarily large amount of time he must exit the region, the probability he ends up on $x + y = 4$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by Andrew Wen

22 Let p and q be positive integers such that p is a prime, p divides $q-1$, and $p+q$ divides $p^2+2020q^2$. Find the sum of the possible values of p.

Proposed by Andy Xu

23 Let P be a point chosen on the interior of side $\overline{BC}$ of triangle $\triangle ABC$ with side lengths $\overline{AB} = 10, \overline{BC} = 10, \overline{AC} = 12$. If X and Y are the feet of the perpendiculars from P to the sides AB and AC, then the minimum possible value of $PX^2 + PY^2$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Proposed by Andrew Wen

24 Freddy the Frog is situated at 1 on an infinitely long number line. On day n, where $n \geq 1$, Freddy can choose to hop 1 step to the right, stay where he is, or hop k steps to the left, where k is an integer at most $n + 1$. After day 5, how many sequences of moves are there such that Freddy has landed on at least one negative number?

Proposed by Andy Xu