Math Open At Andover - Team Round 2018-19,2021-23 plus Theme Round 2020

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– Team Round

2018.1 In $\triangle ABC$, $AB = 3$, $BC = 5$, and $CA = 6$. Points D and E are chosen such that $ACDE$ is a square which does not overlap with $\triangle ABC$. The length of BD can be expressed in the form $\sqrt{m + n\sqrt{p}}$, where m, n, and p are positive integers and p is not divisible by the square of a prime. Determine the value of $m + n + p$.

2018.2 If $x > 0$ and $x^2 + \frac{1}{x^2} = 14$, find $x^5 + \frac{1}{x^5}$.

2018.3 Let BE and CF be altitudes in triangle ABC. If $AE = 24$, $EC = 60$, and $BF = 31$, determine AF.

2018.4 Michael and Andrew are playing the game Bust, which is played as follows: Michael chooses a positive integer less than or equal to 99, and writes it on the board. Andrew then makes a move, which consists of him choosing a positive integer less than or equal to 8 and increasing the integer on the board by the integer he chose. Play then alternates in this manner, with each person making exactly one move, until the integer on the board becomes greater than or equal to 100. The person who made the last move loses. Let S be the sum of all numbers for which Michael could choose initially and win with both people playing optimally. Find S.

2018.5 Mr. DoBa likes to listen to music occasionally while he does his math homework. When he listens to classical music, he solves one problem every 3 minutes. When he listens to rap music, however, he only solves one problem every 5 minutes. Mr. DoBa listens to a playlist comprised of 60% classical music and 40% rap music. Each song is exactly 4 minutes long. Suppose that the expected number of problems he solves in an hour does not depend on whether or not Mr. DoBa is listening to music at any given moment, and let m the average number of problems Mr. DoBa solves per minute when he is not listening to music. Determine the value of $1000m$.

2018.6 Consider an $m \times n$ grid of unit squares. Let R be the total number of rectangles of any size, and let S be the total number of squares of any size. Assume that the sides of the rectangles and squares are parallel to the sides of the $m \times n$ grid. If $\frac{R}{S} = \frac{759}{50}$, then determine mn.

2018.7 For a positive integer k, define the k-*pop* of a positive integer n as the infinite sequence of integers $a_1, a_2, ...$ such that $a_1 = n$ and

$$a_{i+1} = \left\lfloor \frac{a_i}{k} \right\rfloor, i = 1, 2, ..$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Furthermore, define a positive integer m to be k*-pop avoiding* if k does not divide any nonzero term in the k-pop of m. For example, 14 is 3-pop avoiding because 3 does not divide any nonzero term in the 3-pop of 14, which is $14, 4, 1, 0, 0, \ldots$. Suppose that the number of positive integers less than 13^{2018} which are 13-pop avoiding is equal to N. What is the remainder when N is divided by 1000?

2018.8 Suppose that k and x are positive integers such that

$$\frac{k}{2} = \left(\sqrt{1+\frac{\sqrt{3}}{2}}\right)^x + \left(\sqrt{1-\frac{\sqrt{3}}{2}}\right)^x.$$

Find the sum of all possible values of k

2018.9 Quadrilateral $ABCD$ with $AC = 800$ is inscribed in a circle, and E, W, X, Y, Z are the midpoints of segments BD, AB, BC, CD, DA, respectively. If the circumcenters of EWZ and EXY are O_1 and O_2, respectively, determine O_1O_2.

2018.10 Vincent is playing a game with Evil Bill. The game uses an infinite number of red balls, an infinite number of green balls, and a very large bag. Vincent first picks two nonnegative integers g and k such that $g < k \le 2016$, and Evil Bill places g green balls and $2016 - g$ red balls in the bag, so that there is a total of 2016 balls in the bag. Vincent then picks a ball of either color and places it in the bag. Evil Bill then inspects the bag. If the ratio of green balls to total balls in the bag is ever exactly $\frac{k}{2016}$, then Evil Bill wins. If the ratio of green balls to total balls is greater than $\frac{k}{2016}$, then Vincent wins. Otherwise, Vincent and Evil Bill repeat the previous two actions (Vincent picks a ball and Evil Bill inspects the bag). If S is the sum of all possible values of k that Vincent could choose and be able to win, determine the largest prime factor of S.

2019.1 Jeffrey stands on a straight horizontal bridge that measures 20000 meters across. He wishes to place a pole vertically at the center of the bridge so that the sum of the distances from the top of the pole to the two ends of the bridge is 20001 meters. To the nearest meter, how long of a pole does Jeffrey need?

2019.2 The lengths of the two legs of a right triangle are the two distinct roots of the quadratic $x^2 - 36x + 70$. What is the length of the triangle's hypotenuse?

2019.3 For how many ordered pairs of positive integers (a, b) such that $a \le 50$ is it true that $x^2 - ax + b$ has integer roots?

2019.4 Brandon wants to split his orchestra of 20 violins, 15 violas, 10 cellos, and 5 basses into three distinguishable groups, where all of the players of each instrument are indistinguishable. He wants each group to have at least one of each instrument and for each group to have more violins than violas, more violas than cellos, and more cellos than basses. How many ways are there for Brandon to split his orchestra following these conditions?

2019.5 Let ABC be a triangle with $AB = AC = 10$ and $BC = 12$. Define ℓ_A as the line through A perpendicular to $\overline{AB}$. Similarly, ℓ_B is the line through B perpendicular to $\overline{BC}$ and ℓ_C is the line through C perpendicular to $\overline{CA}$. These three lines ℓ_A, ℓ_B, ℓ_C form a triangle with perimeter m/n for relatively prime positive integers m and n. Find $m + n$.

2019.6 Let $f(x, y) = \left\lfloor \frac{5x}{2y} \right\rfloor + \left\lceil \frac{5y}{2x} \right\rceil$. Suppose x, y are chosen independently uniformly at random from the interval $(0, 1]$. Let p be the probability that $f(x, y) < 6$. If p can be expressed in the form m/n for relatively prime positive integers m and n, compute $m + n$.

(Note: $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x and $\lceil x \rceil$ is defined as the least integer greater than or equal to$x$.)

2019.7 Suppose ABC is a triangle inscribed in circle ω . Let A' be the point on ω so that AA' is a diameter, and let G be the centroid of ABC. Given that $AB = 13$, $BC = 14$, and $CA = 15$, let x be the area of triangle AGA' . If x can be expressed in the form m/n , where m and n are relatively prime positive integers, compute $100n + m$.

2019.8 Suppose that

$$\frac{(\sqrt{2})^5 + 1}{\sqrt{2} + 1} \times \frac{2^5 + 1}{2 + 1} \times \frac{4^5 + 1}{4 + 1} \times \frac{16^5 + 1}{16 + 1} = \frac{m}{7 + 3\sqrt{2}}$$

for some integer m. How many 0's are in the binary representation of m? (For example, the number $20 = 10100_2$ has three 0's in its binary representation.)

2019.9 Jonathan finds all ordered triples (a, b, c) of positive integers such that $abc = 720$. For each ordered triple, he writes their sum $a + b + c$ on the board. (Numbers may appear more than once.) What is the sum of all the numbers written on the board?

2019.10 Let S be the set of all four digit palindromes (a palindrome is a number that reads the same forwards and backwards). The average value of $|m - n|$ over all ordered pairs (m, n), where m and n are (not necessarily distinct) elements of S, is equal to p/q , for relatively prime positive integers p and q. Find $p + q$.

2021.1 The value of

$$\frac{1}{20} - \frac{1}{21} + \frac{1}{20 \times 21}$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by Nathan Xiong

2021.2 Four students Alice, Bob, Charlie, and Diana want to arrange themselves in a line such that Alice is at either end of the line, i.e., she is not in between two students. In how many ways can the students do this?

Proposed by Nathan Xiong

2021.3 For two real numbers x and y, let $x \circ y = \frac{xy}{x+y}$. The value of

$$1 \circ (2 \circ (3 \circ (4 \circ 5)))$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

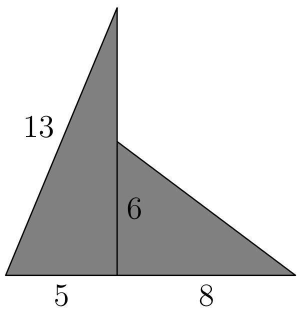
Proposed by Nathan Xiong

2021.4 Compute the number of ordered triples (x, y, z) of integers satisfying

$$x^2 + y^2 + z^2 = 9.$$

Proposed by Nathan Xiong

2021.5 Two right triangles are placed next to each other to form a quadrilateral as shown. What is the perimeter of the quadrilateral?



Proposed by Nathan Xiong

2021.6 Find the sum of all two-digit prime numbers whose digits are also both prime numbers.

Proposed by Nathan Xiong

2021.7 Compute the number of ordered pairs (a, b) of positive integers satisfying $a^b = 2^{100}$.

Proposed by Nathan Xiong

2021.8 Evaluate

$$2^7 \times 3^0 + 2^6 \times 3^1 + 2^5 \times 3^2 + \cdots + 2^0 \times 3^7.$$

Proposed by Nathan Xiong

2021.9 Mr. DoBa has a bag of markers. There are 2 blue, 3 red, 4 green, and 5 yellow markers. Mr. DoBa randomly takes out two markers from the bag. The probability that these two markers are different colors can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Proposed by Raina Yang

2021.10 For how many nonempty subsets $S \subseteq \{1, 2, \ldots, 10\}$ is the sum of all elements in S even?

Proposed by Andrew Wen

2021.11 Find the product of all possible real values for k such that the system of equations

$$x^2 + y^2 = 80$$

$$x^2 + y^2 = k + 2x - 8y$$

has exactly one real solution (x, y).

Proposed by Nathan Xiong

2021.12 Let $\triangle ABC$ have $AB = 9$ and $AC = 10$. A semicircle is inscribed in $\triangle ABC$ with its center on segment BC such that it is tangent AB at point D and AC at point E. If $AD = 2DB$ and r is the radius of the semicircle, r^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Proposed by Andy Xu

2021.13 Bob has 30 identical unit cubes. He can join two cubes together by gluing a face on one cube to a face on the other cube. He must join all the cubes together into one connected solid. Over all possible solids that Bob can build, what is the largest possible surface area of the solid?

Proposed by Nathan Xiong

2021.14 Evaluate

$$\left\lfloor \frac{1 \times 5}{7} \right\rfloor + \left\lfloor \frac{2 \times 5}{7} \right\rfloor + \left\lfloor \frac{3 \times 5}{7} \right\rfloor + \cdots + \left\lfloor \frac{100 \times 5}{7} \right\rfloor.$$

Proposed by Nathan Xiong

2021.15 Consider the polynomial

$$P(x) = x^3 + 3x^2 + 6x + 10.$$

Let its three roots be a, b, c. Define $Q(x)$ to be the monic cubic polynomial with roots ab, bc, ca. Compute $|Q(1)|$.

Proposed by Nathan Xiong

2021.16 Let $\triangle ABC$ have $\angle ABC = 67^\circ$. Point X is chosen such that $AB = XC$, $\angle XAC = 32^\circ$, and $\angle XCA = 35^\circ$. Compute $\angle BAC$ in degrees.

Proposed by Raina Yang

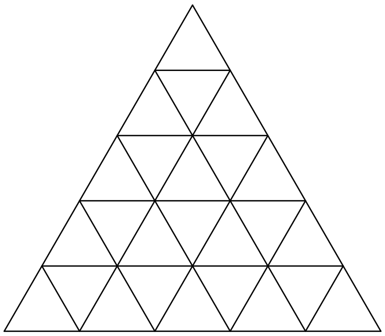
2021.17 Compute the remainder when 10^{2021} is divided by 10101.

Proposed by Nathan Xiong

2021.18 Let $\triangle ABC$ be a triangle with side length $BC = 4\sqrt{6}$. Denote ω as the circumcircle of $\triangle ABC$. Point D lies on ω such that AD is the diameter of ω. Let N be the midpoint of arc BC that contains A. H is the intersection of the altitudes in $\triangle ABC$ and it is given that $HN = HD = 6$. If the area of $\triangle ABC$ can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers with a and c relatively prime and b not divisible by the square of any prime, compute $a + b + c$.

Proposed by Andy Xu

2021.19 Consider the 5 by 5 by 5 equilateral triangular grid as shown:



Ethan chooses two distinct upward-oriented equilateral triangles bounded by the gridlines. The probability that Ethan chooses two triangles that share exactly one vertex can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Proposed by Andrew Wen

2021.20 Compute the sum of all integers x for which there exists an integer y such that

$$x^3 + xy + y^3 = 503.$$

Proposed by Nathan Xiong

2022.1 Consider the 5 by 5 equilateral triangular grid as shown: `https://cdn.artofproblemsolving.com/attachments/1/2/cac43ae24fd4464682a7992e62c99af4acaf8f.png`
How many equilateral triangles are there with sides along the gridlines?

2022.2 While doing her homework for a Momentum Learning class, Valencia draws two intersecting segments $AB = 10$ and $CD = 7$ on a plane. Across all possible configurations of those two segments, determine the maximum possible area of quadrilateral $ACBD$.

2022.3 The area of the figure enclosed by the x-axis, y-axis, and line $7x + 8y = 15$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

2022.4 Angeline flips three fair coins, and if there are any tails, she then flips all coins that landed tails each one more time. The probability that all coins are now heads can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

2022.5 Find the smallest positive integer that is equal to the sum of the product of its digits and the sum of its digits.

2022.6 Define a positive integer n to be *almost-cubic* if it becomes a perfect cube upon concatenating the digit 5. For example, 12 is almost-cubic because $125 = 5^3$. Find the remainder when the sum of all almost-cubic $n < 10^8$ is divided by 1000.

2022.7 A point P is chosen uniformly at random in the interior of triangle ABC with side lengths $AB = 5$, $BC = 12$, $CA = 13$. The probability that a circle with radius $\frac{1}{3}$ centered at P does not intersect the perimeter of ABC can be written as $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

2022.8 Raina the frog is playing a game in a circular pond with six lilypads around its perimeter numbered clockwise from 1 to 6 (so that pad 1 is adjacent to pad 6). She starts at pad 1, and when she is on pad i, she may jump to one of its two adjacent pads, or any pad labeled with j for which $j - i$ is even. How many jump sequences enable Raina to hop to each pad exactly once?

2022.9 Emily has two cups A and B, each of which can hold 400 mL, A initially with 200 mL of water and B initially with 300 mL of water. During a round, she chooses the cup with more water (randomly picking if they have the same amount), drinks half of the water in the chosen cup, then pours the remaining half into the other cup and refills the chosen cup to back to half full. If Emily goes for 20 rounds, how much water does she drink, to the nearest integer?

2022.10 Three integers A, B, C are written on a whiteboard. Every move, Mr. Doba can either subtract 1 from all numbers on the board, or choose two numbers on the board and subtract 1 from both of them whilst leaving the third untouched. For how many ordered triples (A, B, C) with $1 \leq A < B < C \leq 20$ is it possible for Mr. Doba to turn all three of the numbers on the board to 0?

2022.11 Let a *triplet* be some set of three distinct pairwise parallel lines. 20 triplets are drawn on a plane. Find the maximum number of regions these 60 lines can divide the plane into.

2022.12 Triangle ABC has circumcircle ω where B' is the point diametrically opposite B and C' is the point diametrically opposite C. Given $B'C'$ passes through the midpoint of AB, if $AC' = 3$ and $BC = 7$, find AB'^2..

2022.13 Determine the number of distinct positive real solutions to

$$\lfloor x \rfloor^{\{x\}} = \frac{1}{2022}x^2$$

.

Note: $\lfloor x \rfloor$ is known as the floor function, which returns the greatest integer less than or equal to x. Furthermore, $\{x\}$ is defined as $x - \lfloor x \rfloor$.

2022.14 Find the greatest prime number p for which there exists a prime number q such that p divides $4^q + 1$ and q divides $4^p + 1$.

2022.15 Let I_B, I_C be the B, C-excenters of triangle ABC, respectively. Let O be the circumcenter of ABC. If BI_B is perpendicular to AO, $AI_C = 3$ and $AC = 4\sqrt{2}$, then AB^2 can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Note: In triangle $\triangle\ ABC$, the A-excenter is the intersection of the exterior angle bisectors of $\angle ABC$ and $\angle ACB$. The B-excenter and C-excenter are defined similarly.

2023.1 Find the last two digits of $2023 + 202^3 + 20^{23}$.

Proposed by Anthony Yang

2023.2 Let $ABCD$ be a square with side length 6. Let E be a point on the perimeter of $ABCD$ such that the area of $\triangle AEB$ is $\frac{1}{6}$ the area of $ABCD$. Find the maximum possible value of CE^2.

Proposed by Anthony Yang

2023.3 After the final exam, Mr. Liang asked each of his 17 students to guess the average final exam score. David, a very smart student, received a 100 and guessed the average would be 97. Each of the other 16 students guessed $30 + \frac{n}{2}$ where n was that student's score. If the average of the final exam scores was the same as the average of the guesses, what was the average score on the final exam?

Proposed by Eric Wang

2023.4 Andy has 4 coins c_1, c_2, c_3, c_4 such that the probability that coin c_i with $1 \leq i \leq 4$ lands tails is $\frac{1}{2^i}$. Andy flips each coin exactly once. The probability that only one coin lands on heads can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Proposed by Anthony Yang

2023.5 Angeline starts with a 6-digit number and she moves the last digit to the front. For example, if she originally had 100823 she ends up with 310082. Given that her new number is 4 times her original number, find the smallest possible value of her original number.

Proposed by Angeline Zhao

2023.6 Call a set of integers *unpredictable* if no four elements in the set form an arithmetic sequence. How many unordered *unpredictable* sets of five distinct positive integers $\{a, b, c, d, e\}$ exist such that all elements are strictly less than 12?

Proposed by Anthony Yang

2023.7 In a cube, let M be the midpoint of one of the segments. Choose two vertices of the cube, A and B. What is the number of distinct possible triangles $\triangle AMB$ up to congruency?

Proposed by Harry Kim

2023.8 Two consecutive positive integers n and $n+1$ have the property that they both have 6 divisors but a different number of distinct prime factors. Find the sum of the possible values of n.

Proposed by Harry Kim

2023.9 Let $ABCDEF$ be an equiangular hexagon. Let P be the point that is a distance of 6 from BC, DE, and FA. If the distances from P to AB, CD, and EF are 8, 11, and 5 respectively, find $(DE-AB)^2$.

Proposed by Andy Xu

2023.10 Let S be the set of lattice points (a, b) in the coordinate plane such that $1 \leq a \leq 30$ and $1 \leq b \leq 30$. What is the maximum number of lattice points in S such that no four points form a square of side length 2?

Proposed by Harry Kim

2023.11 Let the quadratic $P(x) = x^2 + 5x + 1$. Two distinct real numbers a, b satisfy

$$P(a+b) = ab$$

$$P(ab) = a+b$$

Find the sum of all possible values of a^2.

Proposed by Harry Kim

2023.12 Let N be the number of 105-digit positive integers that contain the digit 1 an odd number of times. Find the remainder when N is divided by 1000.

Proposed by Harry Kim

2023.13 If real numbers x, y, and z satisfy $x^2 - yz = 1$ and $y^2 - xz = 4$ such that $|x+y+z|$ is minimized, then $z^2 - xy$ can be expressed in the form $\sqrt{a} - b$ where a and b are positive integers. Find $a+b$.

Proposed by Andy Xu

2023.14 For a positive integer n, let function $f(n)$ denote the number of positive integers $a \le n$ such that $\gcd(a,n) = \gcd(a+1,n) = 1$. Find the sum of all n such that $f(n) = 15$.

Proposed by Harry Kim

2023.15 Triangle ABC has circumcircle ω. Let D be the foot of the altitude from A to BC and let AD intersect ω at $E \neq A$. Let M be the midpoint of AD. If $\angle BMC = 90^\circ$, $AB = 9$ and $AE = 10$, the area of $\triangle ABC$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers and b is square-free. Find $a+b+c$.

Proposed by Andy Xu

– 2020 Theme Round

– Optimization

TO1 The number 2020 has three different prime factors. What is their sum?

TO2 The Den has two deals on chicken wings. The first deal is 4 chicken wings for 3 dollars, and the second deal is 11 chicken wings for 8 dollars. If Jeremy has 18 dollars, what is the largest number of chicken wings he can buy?

TO3 Consider the addition

$$\begin{array}{cccc} & \text{O} & \text{N} & \text{E} \\ + & \text{T} & \text{W} & \text{O} \\ \hline \text{F} & \text{O} & \text{U} & \text{R} \end{array}$$

where different letters represent different nonzero digits. What is the smallest possible value of the four-digit number $FOUR$?

TO4 Over all real numbers x, let k be the minimum possible value of the expression

$$\sqrt{x^2+9} + \sqrt{x^2-6x+45}.$$

Determine k^2.

TO5 For a real number x, the minimum value of the expression

$$\frac{2x^2+x-3}{x^2-2x+3}$$

can be written in the form $\frac{a-\sqrt{b}}{c}$, where a, b, and c are positive integers such that a and c are relatively prime. Find $a+b+c$

Relay [i]Each problem in this section will depend on the previous one!
The values A, B, C, and D refer to the answers to problems $1, 2, 3$, and 4, respectively.[/i]

TR1. The number 2020 has three different prime factors. What is their sum?

TR2. Let A be the answer to the previous problem. SupposeABC is a triangle with $AB = 81$, $BC = A$, and $\angle ABC = 90^o$. Let D be the midpoint of BC. The perimeter of $\triangle CAD$ can be written as $x+y\sqrt{z}$, where x, y, and z are positive integers and z is not divisible by the square of any prime. What is $x+y$?

TR3. Let B the answer to the previous problem. What is the unique real value of k such that the parabola $y = Bx^2 + k$ and the line $y = kx + B$ are tangent?

TR4. Let C be the answer to the previous problem. How many ordered triples of positive integers (a, b, c) are there such that $gcd(a, b) = gcd(b, c) = 1$ and $abc = C$?

TR5. Let D be the answer to the previous problem. Let $ABCD$ be a square with side length D and circumcircle ω. Denote points C' and D' as the reflections over line AB of C and D respectively. Let P and Q be the points on ω, with$A$ and P on opposite sides of line BC and B and Q on opposite sides of line AD, such that lines $C'P$ and $D'Q$ are both tangent to ω. If the lines AP and BQ intersect at T, what is the area of $\triangle CDT$?

PS. You had better use hide for answers.
