Harvard-MIT Mathematics Tournament, November - Team Round

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2009.13 Down the In finite Corridor

Consider an isosceles triangle T with base 10 and height 12. Defi ne a sequence ω_1, ω_2,...of circles such that ω_1 is the incircle of T and ω_{i+1} is tangent to ω_i and both legs of the isosceles triangle for $i > 1$.

p1. Find the radius of ω_1.

p2. Find the ratio of the radius of ω_{i+1} to the radius of ω_i.

p3. Find the total area contained in all the circles.

2009.48 Bouncy Balls

In the following problems, you will consider the trajectories of balls moving and bouncing off of the boundaries of various containers. The balls are small enough that you can treat them as points. Let us suppose that a ball starts at a point X, strikes a boundary (indicated by the line segment AB) at Y , and then continues, moving along the ray YZ. Balls always bounce in such a way that $\angle XYA = \angle BYZ$. This is indicated in the above diagram.
https://cdn.artofproblemsolving.com/attachments/4/6/42ad28823d839f804d618a1331db43a9ebdca
png

Balls bounce off of boundaries in the same way light reflects off of mirrors - if the ball hits the boundary at point P, the trajectory after P is the reflection of the trajectory before P through the perpendicular to the boundary at P.
A ball inside a rectangular container of width 7 and height 12 is launched from the lower-left vertex of the container. It first strikes the right side of the container after traveling a distance of $\sqrt{53}$ (and strikes no other sides between its launch and its impact with the right side).

p4. Find the height at which the ball first contacts the right side.

p5. How many times does the ball bounce before it returns to a vertex? (The final contact with a vertex does not count as a bounce.)

Now a ball is launched from a vertex of an equilateral triangle with side length 5. It strikes the opposite side after traveling a distance of $\sqrt{19}$.

p6. Find the distance from the ball's point of rst contact with a wall to the nearest vertex.

p7. How many times does the ball bounce before it returns to a vertex? (The final contact with a vertex does not count as a bounce.)

In this final problem, a ball is again launched from the vertex of an equilateral triangle with side length 5.

p8. In how many ways can the ball be launched so that it will return again to a vertex for the first time after 2009 bounces?

2009.911 Super Mario 64!

Mario is once again on a quest to save Princess Peach. Mario enters Peach's castle and fi nds himself in a room with 4 doors. This room is the fi rst in a sequence of 2 indistinugishable rooms. In each room, 1 door leads to the next room in the sequence (or, for the second room, into Bowser's level), while the other 3 doors lead to the fi rst room.

p9. Suppose that in every room, Mario randomly picks a door to walk through. What is the expected number of doors (not including Mario's initial entrance to the fi rst room) through which Mario will pass before he reaches Bowser's level?

p10. Suppose that instead there are 6 rooms with 4 doors. In each room, 1 door leads to the next room in the sequence (or, for the last room, Bowser's level), while the other 3 doors lead to the first room. Now what is the expected number of doors through which Mario will pass before he reaches Bowser's level?

p11. In general, if there are d doors in every room (but still only 1 correct door) and r rooms, the last of which leads into Bowser's level, what is the expected number of doors through which Mario will pass before he reaches Bowser's level?

2010.14 Polyhedron Hopping

p1. Travis is hopping around on the vertices of a cube. Each minute he hops from the vertex he's currently on to the other vertex of an edge that he is next to. After four minutes, what is the probability that he is back where he started?

p2. In terms of k, for $k > 0$ how likely is he to be back where he started after $2k$ minutes?

p3. While Travis is having fun on cubes, Sherry is hopping in the same manner on an octahedron. An octahedron has six vertices and eight regular triangular faces. After ve minutes, how likely is Sherry to be one edge away from where she started?

p4. In terms of k, for $k > 0$, how likely is it that after k minutes Sherry is at the vertex opposite the vertex where she started?

2010.5 Circle O has chord AB. A circle is tangent to O at T and tangent toAB at X such that $AX = 2XB$. What is $\frac{AT}{BT}$?

2010.6 AB is a diameter of circle O. X is a point on AB such that $AX = 3BX$. Distinct circles ω_1 and ω_2 are tangent to O at T_1 and T_2 and to AB at X. The lines T_1X and T_2X intersect O again at S_1 and S_2. What is the ratio $\frac{T_1T_2}{S_1S_2}$?

2010.7 ABC is a right triangle with $\angle A = 30^o$ and circumcircle O. Circles ω_1, ω_2, and ω_3 lie outside ABC and are tangent to O at T_1, T_2, and T_3 respectively and to AB, BC, and CA at S_1, S_2, and S_3, respectively. Lines T_1S_1, T_2S_2, and T_3S_3 intersect O again at A', B', and C', respectively. What is the ratio of the area of $A'B'C'$ to the area of ABC?

2010.810 Linear? What's The Problem?

A function $f(x_1, x_2, ..., x_n)$ is said to be linear in each of its variables if it is a polynomial such that no variable appears with power higher than one in any term. For example, $1+x+xy$ is linear in x and y, but $1 + x^2$ is not. Similarly, $2x + 3yz$ is linear in x, y, and z, but xyz^2 is not.

p8. A function $f(x, y)$ is linear in x and in y. $f(x, y) = \frac{1}{xy}$ for $x, y \in \{3, 4\}$. What is $f(5, 5)$?

p9. A function $f(x, y, z)$ is linear in x, y, and z such that $f(x, y, z) = \frac{1}{xyz}$ for $x, y, z \in \{3, 4\}$. What is $f(5, 5, 5)$?

p10. A function $f(x_1, x_2, ..., x_n)$ is linear in each of the x_i and $f(x_1, x_2, ..., x_n) = \frac{1}{x_1x_2...x_n}$ when $x_i \in \{3, 4\}$ for all i. In terms of n, what is $f(5, 5, ..., 5)$?

2011.1 Find the number of positive integers x less than 100 for which

$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$

is prime.

2011.2 Determine the set of all real numbers p for which the polynomial $Q(x) = x^3 + px^2 - px - 1$ has three distinct real roots.

2011.3 Find the sum of the coefficients of the polynomial $P(x) = x^4 - 29x^3 + ax^2 + bx + c$, given that $P(5) = 11$, $P(11) = 17$, and $P(17) = 23$.

2011.4 Determine the number of quadratic polynomials $P(x) = p_1x^2 + p_2x - p_3$, where p_1, p_2, p_3 are not necessarily distinct (positive) prime numbers less than 50, whose roots are distinct rational numbers.

2011.5 Sixteen wooden Cs are placed in a 4-by-4 grid, all with the same orientation, and each is to be colored either red or blue. A quadrant operation on the grid consists of choosing one of the four two-by-two subgrids of Cs found at the corners of the grid and moving each C in the subgrid to the adjacent square in the subgrid that is 90 degrees away in the clockwise direction, without changing the orientation of the C. Given that two colorings are the considered same if and only

if one can be obtained from the other by a series of quadrant operations, determine the number of distinct colorings of the Cs.
https://cdn.artofproblemsolving.com/attachments/a/9/1e59dce4d33374960953c0c99343eef807a5c png

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2011.6 Ten Cs are written in a row. Some Cs are upper-case and some are lower-case, and each is written in one of two colors, green and yellow. It is given that there is at least one lower-case C, at least one green C, and at least one C that is both upper-case and yellow. Furthermore, no lower-case C can be followed by an upper-case C, and no yellow C can be followed by a green C. In how many ways can the Cs be written?

2011.7 Julia is learning how to write the letter C. She has 6 differently-colored crayons, and wants to write Cc Cc Cc Cc Cc. In how many ways can she write the ten Cs, in such a way that each upper case C is a different color, each lower case C is a different color, and in each pair the upper case C and lower case C are different colors?

2011.8 Let $G, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, B_5$ be ten points on a circle such that $GA_1A_2A_3A_4$ is a regular pentagon and $GB-1B_2B_3B_4B_5$ is a regular hexagon, and B_1 lies on minor arc GA_1. Let B_5B_3 intersect B_1A_2 at G_1, and let B_5A_3 intersect GB_3 at G_2. Determine the degree measure of $\angle GG2G_1$.

2011.9 Let ABC be a triangle with $AB = 9$, $BC = 10$, and $CA = 17$. Let B' be the reflection of the point B over the line CA. Let G be the centroid of triangle ABC, and let G' be the centroid of triangle $AB'C$. Determine the length of segment GG'.

2011.10 Let $G_1G_2G_3$ be a triangle with $G_1G_2 = 7$, $G_2G_3 = 13$, and $G_3G_1 = 15$. Let G_4 be a point outside triangle $G_1G_2G_3$ so that ray $\overrightarrow{G_1G_4}$ cuts through the interior of the triangle, $G_3G_4 = G_4G_2$, and $\angle G_3G_1G_4 = 30^o$. Let G_3G_4 and G_1G_2 meet at G_5. Determine the length of segment G_2G_5.

2012.1 Find the number of integers between 1 and 200 inclusive whose distinct prime divisors sum to 16.
(For example, the sum of the distinct prime divisors of 12 is $2 + 3 = 5$.)

In this section, the word *divisor* is used to refer to a *positive divisor* of an integer.

2012.2 Find the number of ordered triples of divisors (d_1, d_2, d_3) of 360 such that $d_1d_2d_3$ is also a divisor of 360.

In this section, the word *divisor* is used to refer to a *positive divisor* of an integer.

2012.3 Find the largest integer less than 2012 all of whose divisors have at most two 1's in their binary representations.

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2012.4 Let π be a permutation of the numbers from 2 through 2012. Find the largest possible value of

$$\log_2 \pi(2) \cdot \log_3 \pi(3) \ldots \log_{2012} \pi(2012).$$

2012.5 Let π be a randomly chosen permutation of the numbers from 1 through 2012. Find the probability that

$$\pi(\pi(2012)) = 2012.$$

2012.6 Let π be a permutation of the numbers from 1 through 2012. What is the maximum possible number of integers n with $1 \leq n \leq 2011$ such that $\pi(n)$ divides $\pi(n+1)$?

2012.7 Let $A_1 A_2 \ldots A_{100}$ be the vertices of a regular 100-gon. Let π be a randomly chosen permutation of the numbers from 1 through 100. The segments $A_{\pi(1)}A_{\pi(2)}$, $A_{\pi(2)}A_{\pi(3)}$, ... ,$A_{\pi(99)}A_{\pi(100)}$, $A_{\pi(100)}A_{\pi(1)}$ are drawn. Find the expected number of pairs of line segments that intersect at a point in the interior of the 100-gon.

2012.8 ABC is a triangle with $AB = 15$, $BC = 14$, and $CA = 13$. The altitude from A to BC is extended to meet the circumcircle of ABC at D. Find AD.

2012.9 Triangle ABC satisfies $\angle B > \angle C$. Let M be the midpoint of BC, and let the perpendicular bisector of BC meet the circumcircle of $\triangle ABC$ at a point D such that points A, D, C, and B appear on the circle in that order. Given that $\angle ADM = 68^o$ and $\angle DAC = 64^o$, find $\angle B$.

2012.10 Triangle ABC has $AB = 4$, $BC = 5$, and $CA = 6$. Points A', B', C' are such that $B'C'$ is tangent to the circumcircle of ABC at A, $C'A'$ is tangent to the circumcircle at B, and $A'B'$ is tangent to the circumcircle at C. Find the length $B'C'$.

2013.1 Tim the Beaver can make three different types of geometrical figures: squares, regular hexagons, and regular octagons. Tim makes a random sequence F_0, F_1, F_2, F_3, ... of figures as follows: • F_0 is a square. • For every positive integer i, F_i is randomly chosen to be one of the 2 figures distinct from F_{i-1} (each chosen with equal probability $\frac{1}{2}$). • Tim takes 4 seconds to make squares, 6 to make hexagons, and 8 to make octagons. He makes one figure after another, with no breaks in between.
Suppose that exactly 17 seconds after he starts making F_0, Tim is making a figure with n sides. What is the expected value of n?

2013.2 Gary plays the following game with a fair n-sided die whose faces are labeled with the positive integers between 1 and n, inclusive: if $n = 1$, he stops; otherwise he rolls the die, and starts over with a k-sided die, where k is the number his n-sided die lands on. (In particular, if he gets $k = 1$, he will stop rolling the die.) If he starts out with a 6-sided die, what is the expected number of rolls he makes?

2013.3 The digits $1, 2, 3, 4, 5, 6$ are randomly chosen (without replacement) to form the three-digit numbers $M = \overline{ABC}$ and $N = \overline{DEF}$. For example, we could have $M = 413$ and $N = 256$. Find the expected value of $M \cdot N$.

2013.4 Consider triangle ABC with side lengths $AB = 4$, $BC = 7$, and $AC = 8$. Let M be the midpoint of segment AB, and let N be the point on the interior of segment AC that also lies on the circumcircle of triangle MBC. Compute BN.

2013.5 In triangle ABC, $\angle BAC = 60^o$/ Let ω be a circle tangent to segment AB at point D and segment AC at point E. Suppose ω intersects segment BC at points F and G such that$F$ lies in between B and G. Given that $AD = FG = 4$ and $BF = \frac{1}{2}$, find the length of CG.

2013.6 Points A, B, C lie on a circle ω such that BC is a diameter. AB is extended past B to point B' and AC is extended past C to point C' such that line $B'C'$ is parallel to BC and tangent to ω at point D. If $B'D = 4$ and $C'D = 6$, compute BC.

2013.7 In equilateral triangle ABC, a circle ω is drawn such that it is tangent to all three sides of the triangle. A line is drawn from A to point D on segment BC such that AD intersects ω at points E and F. If $EF = 4$ and $AB = 8$, determine $|AE - FD|$.

2013.8 Define the sequence $\{x_i\}_{i\geq 0}$ by $x_0 = x_1 = x_2 = 1$ and $x_k = \frac{x_{k-1}+x_{k-2}+1}{x_{k-3}}$ for $k > 2$. Find x_{2013}.

2013.9 For an integer $n \geq 0$, let $f(n)$ be the smallest possible value of $|x+y|$, where x and y are integers such that $3x - 2y = n$. Evaluate $f(0) + f(1) + f(2) + ... + f(2013)$.

2013.10 Let $\omega = \cos\frac{2\pi}{727} + i \sin\frac{2\pi}{727}$. The imaginary part of the complex number

$$\prod_{k=8}^{13} \left(1 + \omega^{3^{k-1}} + \omega^{2\cdot 3^{k-1}}\right)$$

is equal to $\sin a$ for some angle a between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive. Find a.

2014.1 What is the smallest positive integer n which cannot be written in any of the following forms? • $n = 1 + 2 + ... + k$ for a positive integer k. • $n = p^k$ for a prime number p and integer k. • $n = p + 1$ for a prime number p.

2014.2 Let $f(x) = x^2 + 6x + 7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x.

2014.3 The side lengths of a triangle are distinct positive integers. One of the side lengths is a multiple of 42, and another is a multiple of 72. What is the minimum possible length of the third side?

2014.4 How many ways are there to color the vertices of a triangle red, green, blue, or yellow such that no two vertices have the same color? Rotations and reflections are considered distinct.

2014.5 Let A, B, C, D, E be five points on a circle; some segments are drawn between the points so that each of the $5C2 = 10$ pairs of points is connected by either zero or one segments. Determine the number of sets of segments that can be drawn such that:
• It is possible to travel from any of the five points to any other of the five points along drawn segments.
• It is possible to divide the five points into two nonempty sets S and T such that each segment has one endpoint in S and the other endpoint in T.

2014.6 Find the number of strictly increasing sequences of nonnegative integers with the following properties:
• The first term is 0 and the last term is 12. In particular, the sequence has at least two terms.
• Among any two consecutive terms, exactly one of them is even.

2014.7 Sammy has a wooden board, shaped as a rectangle with length 2^{2014} and height 3^{2014}. The board is divided into a grid of unit squares. A termite starts at either the left or bottom edge of the rectangle, and walks along the gridlines by moving either to the right or upwards, until it reaches an edge opposite the one from which the termite started. Depicted below are two possible paths of the termite.
`https://cdn.artofproblemsolving.com/attachments/3/0/39f3b2aa9c61ff24ffc22b968790f4c61da6f png`
The termite's path dissects the board into two parts. Sammy is surprised to find that he can still arrange the pieces to form a new rectangle not congruent to the original rectangle. This rectangle has perimeter P. How many possible values of P are there?

2014.8 Let H be a regular hexagon with side length one. Peter picks a point P uniformly and at random within H, then draws the largest circle with center P that is contained in H. What is this probability that the radius of this circle is less than $1/2$?

2014.9 How many lines pass through exactly two points in the following hexagonal grid?
`https://cdn.artofproblemsolving.com/attachments/2/e/35741c80d0e0ee0ca56f1297b1e377c8db9e2 png`

2014.10 Let $ABCDEF$ be a convex hexagon with the following properties.
(a) $\overline{AC}$ and $\overline{AE}$ trisect $\angle BAF$.
(b) $\overline{BE} \parallel \overline{CD}$ and $\overline{CF} \parallel \overline{DE}$.
(c) $AB = 2AC = 4AE = 8AF$.
Suppose that quadrilaterals $ACDE$ and $ADEF$ have area 2014 and 1400, respectively. Find the

area of quadrilateral $ABCD$.

2015.1 Triangle ABC is isosceles, and $\angle ABC = x^\circ$. If the sum of the possible measures of $\angle BAC$ is 240°, find x.

2015.2 Bassanio has three red coins, four yellow coins, and five blue coins. At any point, he may give Shylock any two coins of different colors in exchange for one coin of the other color; for example, he may give Shylock one red coin and one blue coin, and receive one yellow coin in return. Bassanio wishes to end with coins that are all the same color, and he wishes to do this while having as many coins as possible. How many coins will he end up with, and what color will they be?

2015.3 Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x, and let $\{x\}$ denote the fractional part of x. For example, $\lfloor \pi \rfloor = 3$, and $\{\pi\} = 0.14159\ldots$, while $\lfloor 100 \rfloor = 100$ and $\{100\} = 0$. If n is the largest solution to the equation $\frac{\lfloor n \rfloor}{n} = \frac{2015}{2016}$, compute $\{n\}$.

2015.4 Call a set of positive integers *good* if there is a partition of it into two sets S and T, such that there do not exist three elements $a, b, c \in S$ such that $a^b = c$ and such that there do not exist three elements $a, b, c \in T$ such that $a^b = c$ (a and b need not be distinct). Find the smallest positive integer n such that the set $\{2, 3, 4, \ldots, n\}$ is *not* good.

2015.5 Kelvin the Frog is trying to hop across a river. The river has 10 lilypads on it, and he must hop on them in a specific order (the order is unknown to Kelvin). If Kelvin hops to the wrong lilypad at any point, he will be thrown back to the wrong side of the river and will have to start over. Assuming Kelvin is infinitely intelligent, what is the minimum number of hops he will need to guarantee reaching the other side?

2015.6 Marcus and four of his relatives are at a party. Each pair of the five people are either *friends* or *enemies*. For any two enemies, there is no person that they are both friends with. In how many ways is this possible?

2015.7 Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD meet at P. Let the area of triangle APB be 24 and let the area of triangle CPD be 25. What is the minimum possible area of quadrilateral $ABCD$?

2015.8 Find **any** quadruple of positive integers (a, b, c, d) satisfying $a^3 + b^4 + c^5 = d^{11}$ and $abc < 10^5$.

2015.9 A graph consists of 6 vertices. For each pair of vertices, a coin is flipped, and an edge connecting the two vertices is drawn if and only if the coin shows heads. Such a graph is $good$ if, starting from any vertex V connected to at least one other vertex, it is possible to draw a path starting and ending at V that traverses each edge exactly once. What is the probability that the graph is good?

2015.10 A number n is bad if there exists some integer c for which $x^x \equiv c \pmod n$ has no integer solutions for x. Find the number of bad integers between 2 and 42 inclusive.

2016.1 Two circles centered at O_1 and O_2 have radii 2 and 3 and are externally tangent at P. The common external tangent of the two circles intersects the line O_1O_2 at Q. What is the length of PQ ?

2016.2 What is the smallest possible perimeter of a triangle whose side lengths are all squares of distinct positive integers?

2016.3 Complex number ω satisfies $\omega^5 = 2$. Find the sum of all possible values of $\omega^4 + \omega^3 + \omega^2 + \omega + 1$.

2016.4 Meghal is playing a game with 2016 rounds $1, 2, ..., 2016$. In round n, two rectangular double-sided mirrors are arranged such that they share a common edge and the angle between the faces is $\frac{2\pi}{n+2}$. Meghal shoots a laser at these mirrors and her score for the round is the number of points on the two mirrors at which the laser beam touches a mirror. What is the maximum possible score Meghal could have after she finishes the game?

2016.5 Allen and Brian are playing a game in which they roll a 6-sided die until one of them wins. Allen wins if two consecutive rolls are equal and at most 3. Brian wins if two consecutive rolls add up to 7 and the latter is at most 3. What is the probability that Allen wins

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2016.6 Let ABC be a triangle with $AB = 5$, $BC = 6$, and $AC = 7$. Let its orthocenter be H and the feet of the altitudes from A, B, C to the opposite sides be D, E, F respectively. Let the line DF intersect the circumcircle of AHF again at X. Find the length of EX.

2016.7 Rachel has two indistinguishable tokens, and places them on the first and second square of a 1×6 grid of squares, She can move the pieces in two ways: • If a token has free square in front of it, then she can move this token one square to the right. • If the square immediately to the right of a token is occupied by the other token, then she can "leapfrog" the first token; she moves the first token two squares to the right, over the other token, so that it is on the square immediately to the right of the other token.
If a token reaches the 6th square, then it cannot move forward any more, and Rachel must move the other one until it reaches the 5th square. How many different sequences of moves for the tokens can Rachel make so that the two tokens end up on the 5th square and the 6th square?

2016.8 Alex has an 20×16 grid of lightbulbs, initially all off. He has 36 switches, one for each row and column. Flipping the switch for the ith row will toggle the state of each lightbulb in the ith row (so that if it were on before, it would be off, and vice versa). Similarly, the switch for the jth column will toggle the state of each bulb in the jth column. Alex makes some (possibly empty) sequence of switch flips, resulting in some configuration of the lightbulbs and their states. How many distinct possible configurations of lightbulbs can Alex achieve with such a sequence?

Two configurations are distinct if there exists a lightbulb that is on in one configuration and off in another.

2016.9 A cylinder with radius 15 and height 16 is inscribed in a sphere. Three congruent smaller spheres of radius x are externally tangent to the base of the cylinder, externally tangent to each other, and internally tangent to the large sphere. What is the value of x?

2016.10 Determine the largest integer n such that there exist monic quadratic polynomials $p_1(x), p_2(x)$, $p_3(x)$ with integer coefficients so that for all integers $i \in [1, n]$ there exists some $j \in [1, 3]$ and $m \in Z$ such that $p_j(m) = i$.

2017.1 A positive integer k is called *powerful* if there are distinct positive integers p, q, r, s, t such that p^2, q^3, r^5, s^7, t^{11} all divide k. Find the smallest powerful integer.

2017.2 How many sequences of integers $(a_1, ..., a_7)$ are there for which $-1 \leq a_i \leq 1$ for every i, and

$$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 + a_6a_7 = 4?$$

2017.3 Michael writes down all the integers between 1 and N inclusive on a piece of paper and discovers that exactly 40% of them have leftmost digit 1. Given that $N > 2017$, find the smallest possible value of N.

2017.4 An equiangular hexagon has side lengths $1, 1, a, 1, 1, a$ in that order. Given that there exists a circle that intersects the hexagon at 12 distinct points, we have $M < a < N$ for some real numbers M and N. Determine the minimum possible value of the ratio $\frac{N}{M}$.

2017.5 Ashwin the frog is traveling on the xy-plane in a series of $2^{2017} - 1$ steps, starting at the origin. At the n^{th} step, if n is odd, then Ashwin jumps one unit to the right. If n is even, then Ashwin jumps m units up, where m is the greatest integer such that 2^m divides n. If Ashwin begins at the origin, what is the area of the polygon bounded by Ashwin's path, the line $x = 2^{2016}$, and the x-axis?

2017.6 Consider five-dimensional Cartesian space $R^5 = \{(x_1, x_2, x_3, x_4, x_5)|x_i \in R\}$, and consider the hyperplanes with the following equations: • $x_i = x_j$ for every $1 \leq i < j \leq 5$, • $x_1 + x_2 + x_3 + x_4 + x_5 = -1$, • $x_1 + x_2 + x_3 + x_4 + x_5 = 0$, • $x_1 + x_2 + x_3 + x_4 + x_5 = 1.$
Into how many regions do these hyperplanes divide R^5 ?

2017.7 There are 12 students in a classroom; 6 of them are Democrats and 6 of them are Republicans. Every hour the students are randomly separated into four groups of three for political debates. If a group contains students from both parties, the minority in the group will change his/her political alignment to that of the majority at the end of the debate. What is the expected amount of time needed for all 12 students to have the same political alignment, in hours?

2017.8 Find the number of quadruples (a, b, c, d) of integers with absolute value at most 5 such that

$$(a^2 + b^2 + c^2 + d^2)^2 = (a + b + c + d)(a - b + c - d)((a - c)^2 + (b - d)^2).$$

2017.9 Let A, B, C, D be points chosen on a circle, in that order. Line BD is reflected over lines AB and DA to obtain lines ℓ_1 and ℓ_2 respectively. If lines ℓ_1, ℓ_2, and AC meet at a common point and if $AB = 4$, $BC = 3$, $CD = 2$, compute the length DA.

2017.10 Yannick has a bicycle lock with a 4-digit passcode whose digits are between 0 and 9 inclusive. (Leading zeroes are allowed.) The dials on the lock is currently set at 0000. To unlock the lock, every second he picks a contiguous set of dials, and increases or decreases all of them by one, until the dials are set to the passcode. For example, after the first second the dials could be set to 1100, 0010, or 9999, but not 0909 or 0190. (The digits on each dial are cyclic, so increasing 9 gives 0, and decreasing 0 gives 9.) Let the complexity of a passcode be the minimum number of seconds he needs to unlock the lock. What is the maximum possible complexity of a passcode, and how many passcodes have this maximum complexity? Express the two answers as an ordered pair.

2018.1 Four standard six-sided dice are rolled. Find the probability that, for each pair of dice, the product of the two numbers rolled on those dice is a multiple of 4.

2018.2 Alice starts with the number 0. She can apply 100 operations on her number. In each operation, she can either add 1 to her number, or square her number. After applying all operations, her score is the minimum distance from her number to any perfect square. What is the maximum score she can attain?

2018.3 For how many positive integers $n \leq 100$ is it true that $10n$ has exactly three times as many positive divisors as n has?

2018.4 Let a and b be real numbers greater than 1 such that $ab = 100$. The maximum possible value of $a^{(\log_{10} b)^2}$ can be written in the form 10^x for some real number x. Find x.

2018.5 Find the sum of all positive integers n such that $1 + 2 + \cdots + n$ divides

$$15\left[(n+1)^2 + (n+2)^2 + \cdots + (2n)^2\right].$$

2018.6 Triangle $\triangle PQR$, with $PQ = PR = 5$ and $QR = 6$, is inscribed in circle ω. Compute the radius of the circle with center on $\overline{QR}$ which is tangent to both ω and $\overline{PQ}$.

2018.7 A 5×5 grid of squares is filled with integers. Call a rectangle *corner-odd* if its sides are grid lines and the sum of the integers in its four corners is an odd number. What is the maximum possible number of corner-odd rectangles within the grid?

Note: A rectangles must have four distinct corners to be considered *corner-odd*; i.e. no $1 \times k$ rectangle can be *corner-odd* for any positive integer k.

2018.8 Tessa has a unit cube, on which each vertex is labeled by a distinct integer between 1 and 8 inclusive. She also has a deck of 8 cards, 4 of which are black and 4 of which are white. At each step she draws a card from the deck, and-if the card is black, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance 1 away from the vertex;-if the card is white, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance $\sqrt{2}$ away from the vertex.When Tessa finishes drawing all cards of the deck, what is the maximum possible value of a number that is on the cube?

2018.9 Let A, B, C be points in that order along a line, such that $AB = 20$ and $BC = 18$. Let ω be a circle of nonzero radius centered at B, and let ℓ_1 and ℓ_2 be tangents to ω through A and C, respectively. Let K be the intersection of ℓ_1 and ℓ_2. Let X lie on segment $\overline{KA}$ and Y lie on segment $\overline{KC}$ such that $XY \| BC$ and XY is tangent to ω. What is the largest possible integer length for XY?

2018.10 David and Evan are playing a game. Evan thinks of a positive integer N between 1 and 59, inclusive, and David tries to guess it. Each time David makes a guess, Evan will tell him whether the guess is greater than, equal to, or less than N. David wants to devise a strategy that will guarantee that he knows N in five guesses. In David's strategy, each guess will be determined only by Evan's responses to any previous guesses (the first guess will always be the same), and David will only guess a number which satisfies each of Evan's responses. How many such strategies are there?

Note: David need not guess N within his five guesses; he just needs to know what N is after five guesses.

2019.1 Each person in Cambridge drinks a (possibly different) 12 ounce mixture of water and apple juice,
where each drink has a positive amount of both liquids. Marc McGovern, the mayor of Cambridge, drinks $\frac{1}{6}$ of the total amount of water drunk and $\frac{1}{8}$ of the total amount of apple juice drunk. How many people are in Cambridge?

2019.2 2019 students are voting on the distribution of N items. For each item, each student submits a vote on who should receive that item, and the person with the most votes receives the item (in case of a tie, no one gets the item). Suppose that no student votes for the same person twice. Compute the maximum possible number of items one student can receive, over all possible values of N and all possible ways of voting.

2019.3 The coefficients of the polynomial $P(x)$ are nonnegative integers, each less than 100. Given that $P(10) = 331633$ and $P(-10) = 273373$, compute $P(1)$.

2019.4 Two players play a game, starting with a pile of N tokens. On each player's turn, they must remove 2^n tokens from the pile for some nonnegative integer n. If a player cannot make a move, they lose. For how many N between 1 and 2019 (inclusive) does the first player have a winning strategy?

2019.5 Compute the sum of all positive real numbers $x \leq 5$ satisfying

$$x = \frac{\lceil x^2 \rceil + \lceil x \rceil \cdot \lfloor x \rfloor}{\lceil x \rceil + \lfloor x \rfloor}$$

2019.6 Let $ABCD$ be an isosceles trapezoid with $AB = 1$, $BC = DA = 5$, $CD = 7$. Let P be the intersection of diagonals AC and BD, and let Q be the foot of the altitude from D to BC. Let PQ intersect AB at R. Compute $\sin \angle RPD$

2019.7 Consider sequences a of the form $a = (a_1, a_2, ..., a_{20})$ such that each term a_i is either 0 or 1. For each such sequence a, we can produce a sequence $b = (b_1, b_2, ..., b_{20})$, where

$$b_i \begin{cases} a_i + a_{i+1} & i = 1 \\ a_{i-1} + a_i + a_{i+1} & 1 < i < 20 \\ a_{i-1} + a_i & i = 20 \end{cases}$$

2019.8 In $\triangle\ ABC$, the external angle bisector of $\angle BAC$ intersects line BC at D. E is a point on ray $\overrightarrow{AC}$ such that $\angle BDE = 2\angle ADB$. If $AB = 10$, $AC = 12$, and $CE = 33$, compute $\frac{DB}{DE}$.

2019.9 Will stands at a point P on the edge of a circular room with perfectly reflective walls. He shines two laser pointers into the room, forming angles of n^o and $(n+1)^o$ with the tangent at P, where n is a positive integer less than 90. The lasers reflect off of the walls, illuminating the points they hit on the walls, until they reach P again. (P is also illuminated at the end.) What is the minimum possible number of illuminated points on the walls of the room?
`https://cdn.artofproblemsolving.com/attachments/a/9/5548d7b34551369d1b69eae682855bcc406f9`
`jpg`

2019.10 A convex 2019-gon $A_1A_2...A_{2019}$ is cut into smaller pieces along its 2019 diagonals of the form A_iA_{i+3} for $1 \leq i \leq 2019$, where $A_{2020} = A_1$, $A_{2021} = A_2$, and $A_{2022} = A_3$. What is the least possible number of resulting pieces?

2020.1 For how many positive integers $n \leq 1000$ does the equation in real numbers $x^{\lfloor x \rfloor} = n$ have a positive solution for x?

2020.2 How many ways are there to arrange the numbers $\{1, 2, 3, 4, 5, 6, 7, 8\}$ in a circle so that every two adjacent elements are relatively prime? Consider rotations and reflections of the same arrangement to be indistinguishable.

2020.3 Let A be the area of the largest semicircle that can be inscribed in a quarter-circle of radius 1. Compute $\frac{120A}{\pi}$.
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2020.4 Marisa has two identical cubical dice labeled with the numbers $\{1, 2, 3, 4, 5, 6\}$. However, the two dice are not fair, meaning that they can land on each face with different probability. Marisa rolls the two dice and calculates their sum. Given that the sum is 2 with probability 0.04, and 12 with probability 0.01, the maximum possible probability of the sum being 7 is p. Compute $\lfloor 100p \rfloor$.

2020.5 For each positive integer n, let an be the smallest nonnegative integer such that there is only one positive integer at most n that is relatively prime to all of n, $n+1$, .. , $n+a_n$. If $n < 100$, compute the largest possible value of $n - a_n$.

2020.6 Regular hexagon $P_1P_2P_3P_4P_5P_6$ has side length 2. For $1 \leq i \leq 6$, let C_i be a unit circle centered at P_i and ℓ_i be one of the internal common tangents of C_i and C_{i+2}, where $C_7 = C_1$ and $C_8 = C_2$. Assume that the lines $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$ bound a regular hexagon. The area of this hexagon can be expressed as $\sqrt{\frac{a}{b}}$, where a and b are relatively prime positive integers. Compute $100a+b$.

2020.7 Roger the ant is traveling on a coordinate plane, starting at $(0, 0)$. Every second, he moves from one lattice point to a different lattice point at distance 1, chosen with equal probability. He will continue to move until he reaches some point P for which he could have reached more quickly had he taken a different route. For example, if he goes from $(0, 0)$ to $(1, 0)$ to $(1, 1)$ to $(1, 2)$ to $(0, 2)$, he stops at because he could have gone from $(0, 0)$ to $(0, 1)$ to $(0, 2)$ in only 2 seconds. The expected number of steps Roger takes before he stops can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

2020.8 Altitudes BE and CF of acute triangle ABC intersect at H. Suppose that the altitudes of triangle EHF concur on line BC. If $AB = 3$ and $AC = 4$, then $BC^2 = \frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

2020.9 Alice and Bob take turns removing balls from a bag containing 10 black balls and 10 white balls, with Alice going first. Alice always removes a black ball if there is one, while Bob removes one of the remaining balls uniformly at random. Once all balls have been removed, the expected number of black balls which Bob has can be expressed as a/b, where a and b are relatively prime positive integers. Compute $100a + b$.

2020.10 Let x and y be non-negative real numbers that sum to 1. Compute the number of ordered pairs

(a, b) with $a, b \in \{0, 1, 2, 3, 4\}$ such that the expression $x^a y^b + y^a x^b$ has maximum value 2^{1-a-b}
.

2021.1 Let $ABCD$ be a parallelogram. Let E be the midpoint of AB and F be the midpoint of CD. Points P and Q are on segments EF and CF, respectively, such that A, P, and Q are collinear. Given that $EP = 5$, $PF = 3$, and $QF = 12$, find CQ.

2021.2 Joey wrote a system of equations on a blackboard, where each of the equations was of the form $a + b = c$ or $a \cdot b = c$ for some variables or integers a, b, c. Then Sean came to the board and erased all of the plus signs and multiplication signs, so that the board reads: $x \quad z = 15$ $x \quad y = 12$
$x \quad x = 36$
If x, y, z are integer solutions to the original system, find the sum of all possible values of $100x + 10y + z$.

2021.3 Suppose m and n are positive integers for which

• the sum of the first m multiples of n is 120, and • the sum of the first m^3 multiples ofn^3 is 4032000.

Determine the sum of the first m^2 multiples of n^2

2021.4 Find the number of 10-digit numbers $\overline{a_1 a_2 ... a_{10}}$ which are multiples of 11 such that the digits are non-increasing from left to right, i.e. $a_i \geq a_{i+1}$ for each $1 \leq i \leq 9$.

2021.5 How many ways are there to place 31 knights in the cells of an 8×8 unit grid so that no two attack one another?

(A knight attacks another knight if the distance between the centers of their cells is exactly $\sqrt{5}$.)

2021.6 The taxicab distance between points (x_1, y_1) and (x_2, y_2) is $|x_2 - x_1| + |y_2 - y_1|$. A regular octagon is positioned in the xy plane so that one of its sides has endpoints $(0, 0)$ and $(1, 0)$. Let S be the set of all points inside the octagon whose taxicab distance from some octagon vertex is at most $2/3$. The area of S can be written as m/n , where m, n are positive integers and $gcd(m, n) = 1$. Find $100m + n$.

2021.7 Let $f(x) = x^3 + 3x - 1$ have roots a, b, c. Given that $\frac{1}{a^3+b^3} + \frac{1}{b^3+c^3} + \frac{1}{c^3+a^3}$ can be written as $\frac{m}{n}$, where m, n are positive integers and $gcd(m, n) = 1$, find $100m + n$.

2021.8 Paul and Sara are playing a game with integers on a whiteboard, with Paul going first. When it is Paul's turn, he can pick any two integers on the board and replace them with their product; when it is Sara's turn, she can pick any two integers on the board and replace them with their sum. Play continues until exactly one integer remains on the board. Paul wins if that integer is odd, and Sara wins if it is even.

Initially, there are 2021 integers on the board, each one sampled uniformly at random from the set $\{0, 1, 2, 3, ..., 2021\}$. Assuming both players play optimally, the probability that Paul wins is m/n , where m, n are positive integers and $gcd(m, n) = 1$. Find the remainder when $m + n$ is divided by 1000.

2021.9 Let N be the smallest positive integer for which $x^2 + x + 1$ divides $166 - \sum_{d|N,d>0} x^d$. Find the remainder when N is divided by 1000.

2021.10 Three faces X, Y, Z of a unit cube share a common vertex. Suppose the projections of X, Y, Z onto a fixed plane P have areas x, y, z, respectively. If $x : y : z = 6 : 10 : 15$, then $x + y + z$ can be written as m/n , where m, n are positive integers and $gcd(m, n) = 1$. Find $100m + n$.
