Harvard-MIT November Tournament, Harvard-MIT Mathematics Tournament November 2018

www.artofproblemsolving.com/community/c2881083

by parmenides51, DemonPlat4, Kagebaka, Derive_Foiler, eashang1

–	Team Round
1	Four standard six-sided dice are rolled. Find the probability that, for each pair of dice, the product of the two numbers rolled on those dice is a multiple of 4.
2	Alice starts with the number 0. She can apply 100 operations on her number. In each operation, she can either add 1 to her number, or square her number. After applying all operations, her score is the minimum distance from her number to any perfect square. What is the maximum score she can attain?
3	For how many positive integers $n \leq 100$ is it true that $10n$ has exactly three times as many positive divisors as n has?
4	Let a and b be real numbers greater than 1 such that $ab = 100$. The maximum possible value of $a^{(\log_{10} b)^2}$ can be written in the form 10^x for some real number x. Find x.
5	Find the sum of all positive integers n such that $1 + 2 + \cdots + n$ divides $$15\left[(n+1)^2 + (n+2)^2 + \cdots + (2n)^2\right].$$
6	Triangle $\triangle PQR$, with $PQ = PR = 5$ and $QR = 6$, is inscribed in circle ω. Compute the radius of the circle with center on $\overline{QR}$ which is tangent to both ω and $\overline{PQ}$.
7	A 5×5 grid of squares is filled with integers. Call a rectangle *corner-odd* if its sides are grid lines and the sum of the integers in its four corners is an odd number. What is the maximum possible number of corner-odd rectangles within the grid? Note: A rectangles must have four distinct corners to be considered *corner-odd*; i.e. no $1 \times k$ rectangle can be *corner-odd* for any positive integer k.
8	Tessa has a unit cube, on which each vertex is labeled by a distinct integer between 1 and 8 inclusive. She also has a deck of 8 cards, 4 of which are black and 4 of which are white. At each step she draws a card from the deck, and-if the card is black, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance 1 away from the vertex;-if the card is white, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance $\sqrt{2}$ away from the vertex.When

Tessa finishes drawing all cards of the deck, what is the maximum possible value of a number that is on the cube?

---

9 Let A, B, C be points in that order along a line, such that $AB = 20$ and $BC = 18$. Let ω be a circle of nonzero radius centered at B, and let ℓ_1 and ℓ_2 be tangents to ω through A and C, respectively. Let K be the intersection of ℓ_1 and ℓ_2. Let X lie on segment $\overline{KA}$ and Y lie on segment $\overline{KC}$ such that $XY \| BC$ and XY is tangent to ω. What is the largest possible integer length for XY?

---

10 David and Evan are playing a game. Evan thinks of a positive integer N between 1 and 59, inclusive, and David tries to guess it. Each time David makes a guess, Evan will tell him whether the guess is greater than, equal to, or less than N. David wants to devise a strategy that will guarantee that he knows N in five guesses. In David's strategy, each guess will be determined only by Evan's responses to any previous guesses (the first guess will always be the same), and David will only guess a number which satisfies each of Evan's responses. How many such strategies are there?

Note: David need not guess N within his five guesses; he just needs to know what N is after five guesses.

– Theme Round

---

1 Square $CASH$ and regular pentagon $MONEY$ are both inscribed in a circle. Given that they do not share a vertex, how many intersections do these two polygons have?

---

2 Consider the addition problem:
$$\begin{array}{cccccc} & C & A & S & H \\ + & & & M & E \\ \hline O & S & I & D & E \end{array}$$
where each letter represents a base-ten digit, and $C, M, O \neq 0$. (Distinct letters are allowed to represent the same digit.) How many ways are there to assign values to the letters so that the addition problem is true?

---

3 $HOW, BOW,$ and DAH are equilateral triangles in a plane such that $WO = 7$ and $AH = 2$. Given that D, A, B are collinear in that order, find the length of BA.

---

4 I have two cents and Bill has n cents. Bill wants to buy some pencils, which come in two different packages. One package of pencils costs 6 cents for 7 pencils, and the other package of pencils costs a *dime for a dozen* pencils (i.e. 10 cents for 12 pencils). Bill notes that he can spend **all** n of his cents on some combination of pencil packages to get P pencils. However, if I *give my two cents* to Bill, he then notes that he can instead spend **all** $n+2$ of his cents on some combination of pencil packages to get fewer than P pencils. What is the smallest value of n for which this is possible?

Note: Both times Bill must spend **all** of his cents on pencil packages, i.e. have zero cents after either purchase.

---

5 Lil Wayne, the rain god, determines the weather. If Lil Wayne makes it rain on any given day, the probability that he makes it rain the next day is 75%. If Lil Wayne doesn't make it rain on one day, the probability that he makes it rain the next day is 25%. He decides not to make it rain today. Find the smallest positive integer n such that the probability that Lil Wayne *makes it rain* n days from today is greater than 49.9%.

---

6 Farmer James invents a new currency, such that for every positive integer $n \leq 6$, there exists an n-coin worth $n!$ cents. Furthermore, he has exactly n copies of each n-coin. An integer k is said to be *nice* if Farmer James can make k cents using at least one copy of each type of coin. How many positive integers less than 2018 are nice?

---

7 Ben "One Hunna Dolla" Franklin is flying a kite $KITE$ such that IE is the perpendicular bisector of KT. Let IE meet KT at R. The midpoints of KI, IT, TE, EK are $A, N, M, D,$ respectively. Given that $[MAKE] = 18, IT = 10, [RAIN] = 4,$ find $[DIME]$.

Note: $[X]$ denotes the area of the figure X.

---

8 Crisp All, a basketball player, is *dropping dimes* and nickels on a number line. Crisp drops a dime on every positive multiple of 10, and a nickel on every multiple of 5 that is not a multiple of 10. Crisp then starts at 0. Every second, he has a $\frac{2}{3}$ chance of jumping from his current location x to $x+3$, and a $\frac{1}{3}$ chance of jumping from his current location x to $x+7$. When Crisp jumps on either a dime or a nickel, he stops jumping. What is the probability that Crisp *stops on a dime*?

---

9 Circle ω_1 of radius 1 and circle ω_2 of radius 2 are concentric. Godzilla inscribes square $CASH$ in ω_1 and regular pentagon $MONEY$ in ω_2. It then writes down all 20 (not necessarily distinct) distances between a vertex of $CASH$ and a vertex of $MONEY$ and multiplies them all together. What is the maximum possible value of his result?

---

10 One million *bucks* (i.e. one million male deer) are in different cells of a 1000×1000 grid. The left and right edges of the grid are then glued together, and the top and bottom edges of the grid are glued together, so that the grid forms a doughnut-shaped torus. Furthermore, some of the bucks are *honest bucks*, who always tell the truth, and the remaining bucks are *dishonest bucks*, who never tell the truth.
Each of the million *bucks* claims that "at most one of my neighboring bucks is an *honest buck*."
A pair of *neighboring bucks* is said to be *buckaroo* if exactly one of them is an *honest buck* .
What is the minimum possible number of *buckaroo* pairs in the grid?

Note: Two *bucks* are considered to be *neighboring* if their cells (x_1, y_1) and (x_2, y_2) satisfy either: $x_1 = x_2$ and $y_1 - y_2 \equiv \pm 1 \pmod{1000}$, or $x_1 - x_2 \equiv \pm 1 \pmod{1000}$ and $y_1 = y_2$.

– General Round

---

1 What is the largest factor of 130000 that does not contain the digit 0 or 5?

---

2 Twenty-seven players are randomly split into three teams of nine. Given that Zack is on a different team from Mihir and Mihir is on a different team from Andrew, what is the probability that Zack and Andrew are on the same team?

---

3 A square in the *xy*-plane has area *A*, and three of its vertices have *x*-coordinates $2, 0,$ and 18 in some order. Find the sum of all possible values of *A*.

---

4 Find the number of eight-digit positive integers that are multiples of 9 and have all distinct digits.

---

5 Compute the smallest positive integer n for which

$$\sqrt{100+\sqrt{n}}+\sqrt{100-\sqrt{n}}$$

is an integer.

---

6 Call a polygon *normal* if it can be inscribed in a unit circle. How many non-congruent normal polygons are there such that the square of each side length is a positive integer?

---

7 Anders is solving a math problem, and he encounters the expression $\sqrt{15!}$. He attempts to simplify this radical as $a\sqrt{b}$ where a and b are positive integers. The sum of all possible values of ab can be expressed in the form $q \cdot 15!$ for some rational number q. Find q.

---

8 Equilateral triangle ABC has circumcircle Ω. Points D and E are chosen on minor arcs AB and AC of Ω respectively such that $BC = DE$. Given that triangle ABE has area 3 and triangle ACD has area 4, find the area of triangle ABC.

---

9 20 players are playing in a Super Mario Smash Bros. Melee tournament. They are ranked $1-20$, and player n will always beat player m if $n < m$. Out of all possible tournaments where each player plays 18 distinct other players exactly once, one is chosen uniformly at random. Find the expected number of pairs of players that win the same number of games.

---

10 Real numbers $x, y,$ and z are chosen from the interval $[-1, 1]$ independently and uniformly at random. What is the probability that

$$|x|+|y|+|z|+|x+y+z|=|x+y|+|y+z|+|z+x|?$$
