Harvard-MIT November Tournament, Harvard-MIT Mathematics Tournament November 2016

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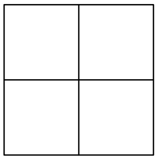
– Guts Round

1-3 1. If five fair coins are flipped simultaneously, what is the probability that at least three of them show heads?

2. How many perfect squares divide 10^{10}?

3. Evaluate $\frac{2016!^2}{2015!2017!}$. Here $n!$ denotes $1 \times 2 \times \ldots \times n$.

4-6 4. A square can be divided into four congruent figures as shown:



For how many n with $1 \leq n \leq 100$ can a unit square be divided into n congruent figures?

5. If $x + 2y - 3z = 7$ and $2x - y + 2z = 6$, determine $8x + y$.

6. Let $ABCD$ be a rectangle, and let E and F be points on segment AB such that $AE = EF = FB$. If CE intersects the line AD at P, and PF intersects BC at Q, determine the ratio of BQ to CQ.

7-9 7. What is the minimum value of the product

$$\prod_{i=1}^{6} \frac{a_i - a_{i+1}}{a_{i+2} - a_{i+3}}$$

given that $(a_1, a_2, a_3, a_4, a_5, a_6)$ is a permutation of $(1, 2, 3, 4, 5, 6)$? (note $a_7 = a_1, a_8 = a_2 \ldots$)

8. Danielle picks a positive integer $1 \leq n \leq 2016$ uniformly at random. What is the probability that gcd$(n, 2015) = 1$?

9. How many 3-element subsets of the set $\{1, 2, 3, ..., 19\}$ have sum of elements divisible by 4?

10-12 10. Michael is playing basketball. He makes 10% of his shots, and gets the ball back after 90% of his missed shots. If he does not get the ball back he stops playing. What is the probability that Michael eventually makes a shot?

11. How many subsets S of the set $\{1, 2, \dots, 10\}$ satisfy the property that, for all $i \in [1, 9]$, either i or $i+1$ (or both) is in S?

12. A positive integer $\overline{ABC}$, where A, B, C are digits, satisfies

$$\overline{ABC} = B^C - A$$

Find $\overline{ABC}$.

13-15 13. How many functions $f : \{0,1\}^3 \to \{0,1\}$ satisfy the property that, for all ordered triples (a_1, a_2, a_3) and (b_1, b_2, b_3) such that $a_i \geq b_i$ for all i, $f(a_1, a_2, a_3) \geq f(b_1, b_2, b_3)$?

14. The very hungry caterpillar lives on the number line. For each non-zero integer i, a fruit sits on the point with coordinate i. The caterpillar moves back and forth; whenever he reaches a point with food, he eats the food, increasing his weight by one pound, and turns around. The caterpillar moves at a speed of 2^{-w} units per day, where w is his weight. If the caterpillar starts off at the origin, weighing zero pounds, and initially moves in the positive x direction, after how many days will he weigh 10 pounds?

15. Let $ABCD$ be an isosceles trapezoid with parallel bases $AB = 1$ and $CD = 2$ and height 1. Find the area of the region containing all points inside $ABCD$ whose projections onto the four sides of the trapezoid lie on the segments formed by AB, BC, CD and DA.

16-18 16. Create a cube C_1 with edge length 1. Take the centers of the faces and connect them to form an octahedron O_1. Take the centers of the octahedron's faces and connect them to form a new cube C_2. Continue this process infinitely. Find the sum of all the surface areas of the cubes and octahedrons.

17. Let $p(x) = x^2 - x + 1$. Let α be a root of $p(p(p(p(x))))$. Find the value of

$$(p(\alpha) - 1)p(\alpha)p(p(\alpha))p(p(p(\alpha))$$

18. An 8 by 8 grid of numbers obeys the following pattern:

1) The first row and first column consist of all 1s.
2) The entry in the ith row and jth column equals the sum of the numbers in the $(i-1)$ by $(j-1)$ sub-grid with row less than i and column less than j.

What is the number in the 8th row and 8th column?

19-21 19. Let S be the set of all positive integers whose prime factorizations only contain powers of the primes 2 and 2017 (1, powers of 2, and powers of 2017 are thus contained in S). Compute $\sum_{s \in S} \frac{1}{s}$.

20. Let $\mathcal{V}$ be the volume enclosed by the graph

$$x^{2016} + y^{2016} + z^2 = 2016$$

Find $\mathcal{V}$ rounded to the nearest multiple of ten.

21. Zlatan has 2017 socks of various colours. He wants to proudly display one sock of each of the
colours, and he counts that there are N ways to select socks from his collection for display. Given this
information, what is the maximum value of N?

22-24 22. Let the function $f : \mathbb{Z} \to \mathbb{Z}$ take only integer inputs and have integer outputs. For any integers x and y, f satisfies

$$f(x) + f(y) = f(x+1) + f(y-1)$$

If $f(2016) = 6102$ and $f(6102) = 2016$, what is $f(1)$?

23. Let d be a randomly chosen divisor of 2016. Find the expected value of

$$\frac{d^2}{d^2 + 2016}$$

24. Consider an infinite grid of equilateral triangles. Each edge (that is, each side of a small triangle) is colored one of N colors. The coloring is done in such a way that any path between any two nonadjecent vertices consists of edges with at least two different colors. What is the smallest possible value of N?

25-27 25. Chris and Paul each rent a different room of a hotel from rooms $1 - 60$. However, the hotel manager mistakes them for one person and gives "Chris Paul" a room with Chris's and Paul's room concatenated. For example, if Chris had 15 and Paul had 9, "Chris Paul" has 159. If there are 360 rooms in the hotel, what is the probability that "Chris Paul" has a valid room?

26. Find the number of ways to choose two nonempty subsets X and Y of $\{1, 2, \ldots, 2001\}$, such that $|Y| = 1001$ and the smallest element of Y is equal to the largest element of X.

27. Let r_1, r_2, r_3, r_4 be the four roots of the polynomial $x^4 - 4x^3 + 8x^2 - 7x + 3$. Find the value of

$$\frac{r_1^2}{r_2^2 + r_3^2 + r_4^2} + \frac{r_2^2}{r_1^2 + r_3^2 + r_4^2} + \frac{r_3^2}{r_1^2 + r_2^2 + r_4^2} + \frac{r_4^2}{r_1^2 + r_2^2 + r_3^2}$$

28-30 28. The numbers $1 - 10$ are written in a circle randomly. Find the expected number of numbers which are at least 2 larger than an adjacent number.

29. We want to design a new chess piece, the American, with the property that (i) the American can never attack itself, and (ii) if an American A_1 attacks another American A_2, then A_2 also

attacks A_1. Let m be the number of squares that an American attacks when placed in the top left corner of an 8 by 8 chessboard. Let n be the maximal number of Americans that can be placed on the 8 by 8 chessboard such that no Americans attack each other, if one American must be in the top left corner. Find the largest possible value of mn.

30. On the blackboard, Amy writes 2017 in base-a to get 133201_a. Betsy notices she can erase a digit from Amy's number and change the base to base-b such that the value of the the number remains the same. Catherine then notices she can erase a digit from Betsy's number and change the base to base-c such that the value still remains the same. Compute, in decimal, $a+b+c$.

31-33 31. Define a number to be an anti-palindrome if, when written in base 3 as $a_n a_{n-1} \dots a_0$, then $a_i + a_{n-i} = 2$ for any $0 \le i \le n$. Find the number of anti-palindromes less than 3^{12} such that no two consecutive digits in base 3 are equal.

32. Let $C_{k,n}$ denote the number of paths on the Cartesian plane along which you can travel from $(0,0)$ to (k,n), given the following rules: 1) You can only travel directly upward or directly rightward 2) You can only change direction at lattice points 3) Each horizontal segment in the path must be at most 99 units long.
Find

$$\sum_{j=0}^{\infty} C_{100j+19,17}$$

33. Camille the snail lives on the surface of a regular dodecahedron. Right now he is on vertex P_1 of the face with vertices P_1, P_2, P_3, P_4, P_5. This face has a perimeter of 5. Camille wants to get to the point on the dodecahedron farthest away from P_1. To do so, he must travel along the surface a distance at least L. What is L^2?

34-36 34. Find the sum of the ages of everyone who wrote a problem for this year's HMMT November contest. If your answer is X and the actual value is Y , your score will be max$(0, 20-|X-Y|)$

35. Find the total number of occurrences of the digits $0, 1 \dots, 9$ in the entire guts round (the official copy). If your
answer is X and the actual value is Y , your score will be max$(0, 20-\frac{|X-Y|}{2})$

36. Find the number of positive integers less than 1000000 which are less than or equal to the sum of their proper divisors. If your answer is X and the actual value is Y, your score will be max$(0, 20-80|1-\frac{X}{Y}|)$ rounded to the nearest integer.

– Team Round

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1 Two circles centered at O_1 and O_2 have radii 2 and 3 and are externally tangent at P. The common external tangent of the two circles intersects the line O_1O_2 at Q. What is the length of PQ ?

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2 What is the smallest possible perimeter of a triangle whose side lengths are all squares of distinct positive integers?

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3 Complex number ω satisfies $\omega^5 = 2$. Find the sum of all possible values of $\omega^4 + \omega^3 + \omega^2 + \omega + 1$.

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4 Meghal is playing a game with 2016 rounds $1, 2, ..., 2016$. In round n, two rectangular double-sided mirrors are arranged such that they share a common edge and the angle between the faces is $\frac{2\pi}{n+2}$. Meghal shoots a laser at these mirrors and her score for the round is the number of points on the two mirrors at which the laser beam touches a mirror. What is the maximum possible score Meghal could have after she finishes the game?

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5 Allen and Brian are playing a game in which they roll a 6-sided die until one of them wins. Allen wins if two consecutive rolls are equal and at most 3. Brian wins if two consecutive rolls add up to 7 and the latter is at most 3. What is the probability that Allen wins

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6 Let ABC be a triangle with $AB = 5$, $BC = 6$, and $AC = 7$. Let its orthocenter be H and the feet of the altitudes from A, B, C to the opposite sides be D, E, F respectively. Let the line DF intersect the circumcircle of AHF again at X. Find the length of EX.

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7 Rachel has two indistinguishable tokens, and places them on the first and second square of a 1×6 grid of squares, She can move the pieces in two ways: • If a token has free square in front of it, then she can move this token one square to the right. • If the square immediately to the right of a token is occupied by the other token, then she can "leapfrog" the first token; she moves the first token two squares to the right, over the other token, so that it is on the square immediately to the right of the other token.
If a token reaches the 6th square, then it cannot move forward any more, and Rachel must move the other one until it reaches the 5th square. How many different sequences of moves for the tokens can Rachel make so that the two tokens end up on the 5th square and the 6th square?

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8 Alex has an 20×16 grid of lightbulbs, initially all off. He has 36 switches, one for each row and column. Flipping the switch for the ith row will toggle the state of each lightbulb in the ith row (so that if it were on before, it would be off, and vice versa). Similarly, the switch for the jth column will toggle the state of each bulb in the jth column. Alex makes some (possibly empty) sequence of switch flips, resulting in some configuration of the lightbulbs and their states. How many distinct possible configurations of lightbulbs can Alex achieve with such a sequence? Two configurations are distinct if there exists a lightbulb that is on in one configuration and off in another.

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9 A cylinder with radius 15 and height 16 is inscribed in a sphere. Three congruent smaller spheres of radius x are externally tangent to the base of the cylinder, externally tangent to each other, and internally tangent to the large sphere. What is the value of x?

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10 Determine the largest integer n such that there exist monic quadratic polynomials $p_1(x)$, $p_2(x)$,

$p_3(x)$ with integer coefficients so that for all integers $i \in [1, n]$ there exists some $j \in [1, 3]$ and $m \in Z$ such that $p_j(m) = i$.

– Theme Round

1 DeAndre Jordan shoots free throws that are worth 1 point each. He makes 40% of his shots. If he takes two shots find the probability that he scores at least 1 point.

2 Point P_1 is located 600 miles West of point P_2. At $7:00$AM a car departs from P_1 and drives East at a speed of 50mph. At $8:00$AM another car departs from P_2 and drives West at a constant speed of x miles per hour. If the cars meet each other exactly halfway between P_1 and P_2, what is the value of x?

3 The three points A, B, C form a triangle. $AB = 4, BC = 5, AC = 6$. Let the angle bisector of $\angle A$ intersect side BC at D. Let the foot of the perpendicular from B to the angle bisector of $\angle A$ be E. Let the line through E parallel to AC meet BC at F. Compute DF.

4 A positive integer is written on each corner of a square such that numbers on opposite vertices are relatively prime while numbers on adjacent vertices are not relatively prime. What is the smallest possible value of the sum of these 4 numbers?

5 Steph Curry is playing the following game and he wins if he has exactly 5 points at some time. Flip a fair coin. If heads, shoot a 3-point shot which is worth 3 points. If tails, shoot a free throw which is worth 1 point. He makes $\frac{1}{2}$ of his 3-point shots and all of his free throws. Find the probability he will win the game. (Note he keeps flipping the coin until he has exactly 5 or goes over 5 points)

6 Let $P_1, P_2, \ldots, P_6$ be points in the complex plane, which are also roots of the equation $x^6 + 6x^3 - 216 = 0$. Given that $P_1P_2P_3P_4P_5P_6$ is a convex hexagon, determine the area of this hexagon.

7 Seven lattice points form a convex heptagon with all sides having distinct lengths. Find the minimum possible value of the sum of the squares of the sides of the heptagon.

8 Let $P_1P_2 \ldots P_8$ be a convex octagon. An integer i is chosen uniformly at random from 1 to 7, inclusive. For each vertex of the octagon, the line between that vertex and the vertex i vertices to the right is painted red. What is the expected number times two red lines intersect at a point that is not one of the vertices, given that no three diagonals are concurrent?

9 The vertices of a regular nonagon are colored such that $1)$ adjacent vertices are different colors and $2)$ if 3 vertices form an equilateral triangle, they are all different colors. Let m be the minimum number of colors needed for a valid coloring, and n be the total number of colorings using m colors. Determine mn. (Assume each vertex is distinguishable.)

10 We have 10 points on a line $A_1, A_2 \dots A_{10}$ in that order. Initially there are n chips on point A_1. Now we are allowed to perform two types of moves. Take two chips on A_i, remove them and place one chip on A_{i+1}, or take two chips on A_{i+1}, remove them, and place a chip on A_{i+2} and A_i . Find the minimum possible value of n such that it is possible to get a chip on A_{10} through a sequence of moves.

– General Round

1 If a and b satisfy the equations $a + \frac{1}{b} = 4$ and $\frac{1}{a} + b = \frac{16}{15}$, determine the product of all possible values of ab.

2 I have five different pairs of socks. Every day for five days, I pick two socks at random without replacement to wear for the day. Find the probability that I wear matching socks on both the third day and the fifth day.

3 Let V be a rectangular prism with integer side lengths. The largest face has area 240 and the smallest face has area 48. A third face has area x, where x is not equal to 48 or 240. What is the sum of all possible values of x?

4 A rectangular pool table has vertices at $(0, 0)(12, 0)(0, 10)$, and $(12, 10)$. There are pockets only in the four corners. A ball is hit from $(0, 0)$ along the line $y = x$ and bounces off several walls before eventually entering a pocket. Find the number of walls that the ball bounces off of before entering a pocket.

5 Let the sequence $\{a_i\}_{i=0}^{\infty}$ be defined by $a_0 = \frac{1}{2}$ and $a_n = 1 + (a_{n-1} - 1)^2$. Find the product

$$\prod_{i=0}^{\infty} a_i = a_0 a_1 a_2 \dots$$

6 The numbers $1, 2 \dots 11$ are arranged in a line from left to right in a random order. It is observed that the middle number is larger than exactly one number to its left. Find the probability that it is larger than exactly one number to its right.

7 Let ABC be a triangle with $AB = 13, BC = 14, CA = 15$. The altitude from A intersects BC at D.
Let ω_1 and ω_2 be the incircles of ABD and ACD, and let the common external tangent of ω_1 and ω_2 (other than BC) intersect AD at E. Compute the length of AE.

8 Let $S = \{1, 2, \dots, 2016\}$, and let f be a randomly chosen bijection from S to itself. Let n be the smallest
positive integer such that $f^{(n)}(1) = 1$, where $f^{(i)}(x) = f(f^{(i-1)}(x))$. What is the expected value of n?

9 Let the sequence a_i be defined as $a_{i+1} = 2^{a_i}$. Find the number of integers $1 \leq n \leq 1000$ such that if $a_0 = n$, then 100 divides $a_{1000} - a_1$.

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10 Quadrilateral $ABCD$ satisfies $AB = 8, BC = 5, CD = 17, DA = 10$. Let E be the intersection of AC and BD. Suppose $BE : ED = 1 : 2$. Find the area of $ABCD$.
