Rice Math Tournament 2003

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– Team Round

– **p1.** What is the ratio of the area of an equilateral triangle to the area of the largest rectangle that can be inscribed inside the triangle?

p2. Define

$$P(x) = x^{12}+12x^{11}+66x^{10}+220x^9+495x^8+792x^7+924x^6+792x^5-159505x^4+220x^3+66x^2+12x+1.$$

Find $\frac{P(19)}{20^4}$.

p3. Four flattened colored cubes are shown below. Each of the cubes' faces has been colored red (R), blue (B), green (G) or yellow (Y). The cubes are stacked on top of each other in numerical order with cube#1 on bottom. The goal of the puzzle is to find an orientation for each cube so that on each of the four visible sides of the stack all four colors appear. Find a solution, and for each side of the stack, list the colors from bottom to top. List the sides in clockwise order.
`https://cdn.artofproblemsolving.com/attachments/0/7/738baf1c399caf842886759cf20478e654620`
`png`

p4. When evaluated, the sum $\sum_{k=1}^{2002}[k \cdot k!]$ is a number that ends with a long series of 9s. How many 9s are at the end of the number?

p5. Find the positive integer n that maximizes the expression $\frac{200003^n}{(n!)^2}$.

p6. Find $11^3 + 12^3 + ... + 100^3$

Hint: Develop a formula for $s(x) = \sum_{i=1}^{x} i^3$ using perhaps $x^3 - (x - 1)^3$.
Also the formulas $\sum_{i=1}^{x} i = \frac{x(x+1)}{2}$ and $\sum_{i=1}^{x} i^2 = \frac{x(x+1)(2x+1)}{6}$ may be helpful.

p7. Six fair 6-sided dice are rolled. What is the probability that the sum of the values on the top faces of the dice is divisible by 7?

p8. Several students take a quiz which has five questions, and each one is worth a point. They are unsure as to how many points they received, but all of them have a reasonable idea about

their scores. Below is a table of what each person thinks is the probability that he or she got each score. Assuming their probabilities are correct, what is the probability that the sum of their scores is exactly 20?
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`png`

p9. Let F_n be the number of ways of completely covering an $3 \times n$ chessboard with n 3×1 dominoes. For example, there are two ways of tiling a 3×3 chessboard with three 3×1 dominoes (all horizontal or all vertical). What is F_{14}?

p10. Two players (Kate and Adam) are playing a variant of Nim. There are 11 sticks in front of the players and they take turns each removing either one or any prime number of sticks. The player who is forced to take the last stick loses. The problem with the game is that if player one (Kate) plays perfectly, she will always win. Give the sum of all the starting moves that lead to a sure win for Kate (assuming each player plays perfectly).

p11. Define $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$. Find all (x, y) such that $(f(x, y))^2 - (g(x, y))^2 = \frac12$ and $f(x, y) \cdot g(x, y) = \frac{\sqrt3}{4}$.

p12. The numerals on digital clocks are made up of seven line segments. When various combinations of them light up different numbers are shown. When a digit on the clock changes, some segments turn on and others turn off. For example, when a 4 changes into a 5 two segments turn on and one segment turns off, for a total of 3 changes. In the usual ordering $1, 2, 3, ..., 0$ there are a sum total of 32 segment changes (including the wrapping around from 0 back to 1). If we can put the digits in any order, what is the fewest total segment changes possible? (As above, include the change from the last digit back to the first. Note that a 1 uses the two vertical segments on the right side, a 6 includes the top segment, and a 9 includes the bottom segment.)

p13. How many solutions are there to $(\cos 10x)(\cos 9x) = \frac12$ for $x \in [0, 2\pi]$?

p14. Find ${2003 \choose 0} + {2003 \choose 4} + {2003 \choose 8} + ...$

Hint: Consider $(1 \pm i)$ and (1 ± 1).

p15. Clue is a board game in which four players attempt to solve a mystery. Mr. Boddy has been killed in one of the nine rooms in his house by one of six people with one of six specific weapons. The board game has 21 cards, each with a person, weapon or room on it. At the beginning of each game, one room card, one person card and one weapon card are randomly chosen and set aside as the solution to the mystery. The remaining cards are all gathered, shuffled, and dealt out to the four players. Note that two players get four cards and two get five cards. Players thus start out knowing only that the cards in their hand are not among the solution cards. Essentially, players then take turns guessing the solution to the mystery. A guess is a listing of a person, weapon and a room. Assume that the player who goes first in the game has only 4 cards. What is the probability that his first guess, the first guess of the game, is exactly right (i.e. he guesses all three hidden cards corrrectly?

PS. You had better use hide for answers.

– Algebra Round

1 1. Let a and b be real numbers and let $a * b = a^2 + ab + 3b + 1$. List all numbers a such that there is no number b such that $a * b = 2$

2 2. What is the smallest positive number k such that there are real number satisfying $a + b = k$ and $ab = k$

3 3. Assume the polynomial $p(x) = x^8 - 12x^6 + 49x^4 - 78x^2 + 42$ has no complex roots. How many negative roots does it have?

4 4. Harry, Hermione, and Ron go to Diagon Alley to buy chocolate frogs. If Harry and Hermione spent one-fourth of their own money, they would spend 3 galleons in total. If Harry and Ron spent one-fifth of their own money, they would spend 24 galleons in total. Everyone has a whole number of galleons, and the number of galleons between the three of them is a multiple of 7. What are all the possible number of galleons that Harry can have?

5 5. If x and y are digits and

$(11xy)_7 = (310x)_5$

Find (x, y)

6 6. Patty is picking peppermints off a tree. They come in two colors: red and white. She picks fewer than 100 total peppermints but at least one of each color. The white flavor is stronger so she prefers red to white. Thus she always picks fewer white peppermints then ten times the number of reds. How many different combinations of peppermints can she go home with?

7 7. Let r_1, r_2, r_3 be the roots of $x^3 - 2x^2 + 4x + 10 = 0$. Compute $(r_1 + 2)(r_2 + 2)(r_3 + 2)$

8 8. Solve for x

$\log_2 \log_4 x + \log_4 \log_2 x = 2$

9 9. Find all real values of x which satsify

$\frac{4}{|x|+1} \geq 1 + |2|x| - 4|$

10 10. Let $b > 0$ and $c > 0$. Suppose the sequence $x_1, x_2, x_3 \cdots$ is defined by

$x_0 = 0$
$x_1 = 1$
$x_{n+2} = bx_{n+1} + cx_n$

Compute $\lim_{x\infty} \frac{x_{n+1}}{x_n}$

– Calculus Round

1 Let $f(x) = \frac{1}{x}$. Compute $f^{(100)}(x)$

2 A windup penguin moves along the x-axis with acceleration given by $a(t) = 2t - 2$ units per second. At $t = 1$ second, the penguin is moving left with a speed of 4 units per second. What is the distance the penguin travels in the first four seconds.

3 The function

$$f(x) = \begin{cases} 2 + x^3 & \text{if } x < 1 \\ 3x & \text{if } x > 1 \end{cases}$$

Is differentiable on $(-1, 2)$ and continuous on $[-1, 2]$ and thus satisfies MVT for derivatives. Find a value of x which is guaranteed to exist by the theorem.

4 Let $f(x) = (x-1)(x-2)(x-3)^2(x-4)(x-5)(x-6)$. Compute $f''(3) - f'(3) + f(3)$

5 Consider a function $f : \mathbb{R} \to \mathbb{R}$. Given that $f(0) = 0$. $\lim_{x\to 0} \frac{f(h)}{h} = 7$, and $f(x+y) = f(x) + f(y) + 3xy$ for all $x, y \in \mathbb{R}$. Compute $f(7)$

6 6. A group of college students get together to write math questions while eating cake. Teens eat $c(t)$ pieces of cake per hour and eat $q(t)$ questions per hour. Additionally $c'(t) = -c(t)$ and $q'(t) = -3c(t)$
At $t = 0$ Teena writes 7 questions and eats 2 pieces of cake per hour. If she ever writes less than 2 questions per hour. The question writing session lasts 24 hours. Does Teena get kicked out? If so, when?

7 Find the derivative of x^{x^x}

8 Let $s > 0$. Consider straight lines drawn from $(0, r)$ to $(s - r, 0)$ for all $0 \leq r \leq s$. We define $f(x)$ as the maximum y value that any of these lines take on at x. Find $f(x)$ for $0 \leq x \leq s$

9 9. We define an hourglass to be a shape composed of two cones of the same size joined at the apex oriend so that the hourglass sits on one of the cone's base. . If this hourglass has a maximal radius of 12 inches and a total height of 10 inches. The sand inside occupies one-fourth of the total volume of the hourglass, and it takes one minute for all the sand from the top cone to drop to the bottom cone. Assuming the sand falls, at a constant rate, what is the rate of change in the height of the sand in the bottom cone 15 seconds before the time runs out?

10 10. Sammy the owl is making a one-eyed Jack o' Lantern. He bought a perfectly spherical pumpkin of radius one foot. He first carves out the inside so that the inner radius is 10 inches. He then drills a perfectly circular hole of radius 1 inch for each eye. He drills straight toward the center of the pumpkin at the equator. What is the total volume in cubic inches of pumpkin he has removed?