Harvard-MIT November Tournament, Harvard-MIT Mathematics Tournament November 2021

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by parmenides51

–	Team
1	Let $ABCD$ be a parallelogram. Let E be the midpoint of AB and F be the midpoint of CD. Points P and Q are on segments EF and CF, respectively, such that A, P, and Q are collinear. Given that $EP = 5$, $PF = 3$, and $QF = 12$, find CQ.
2	Joey wrote a system of equations on a blackboard, where each of the equations was of the form $a + b = c$ or $a \cdot b = c$ for some variables or integers a, b, c. Then Sean came to the board and erased all of the plus signs and multiplication signs, so that the board reads: $x \quad z = 15$ $x \quad y = 12$ $x \quad x = 36$ If x, y, z are integer solutions to the original system, find the sum of all possible values of $100x+10y + z$.
3	Suppose m and n are positive integers for which • the sum of the first m multiples of n is 120, and • the sum of the first m^3 multiples ofn^3 is 4032000. Determine the sum of the first m^2 multiples of n^2
4	Find the number of 10-digit numbers $\overline{a_1a_2...a_{10}}$ which are multiples of 11 such that the digits are non-increasing from left to right, i.e. $a_i \geq a_{i+1}$ for each $1 \leq i \leq 9$.
5	How many ways are there to place 31 knights in the cells of an 8×8 unit grid so that no two attack one another? (A knight attacks another knight if the distance between the centers of their cells is exactly $\sqrt{5}$.)
6	The taxicab distance between points (x_1, y_1) and (x_2, y_2) is $\lvert x_2-x_1\rvert+\lvert y_2-y_1\rvert$. A regular octagon is positioned in the xy plane so that one of its sides has endpoints $(0, 0)$ and $(1, 0)$. Let S be the set of all points inside the octagon whose taxicab distance from some octagon vertex is at most $2/3$. The area of S can be written as m/n , where m, n are positive integers and $gcd(m, n) = 1$. Find $100m + n$.
7	Let $f(x) = x^3 + 3x - 1$ have roots a, b, c. Given that $\frac{1}{a^3+b^3} + \frac{1}{b^3+c^3} + \frac{1}{c^3+a^3}$ can be written as $\frac{m}{n}$, where m, n are positive integers and $gcd(m, n) = 1$, find $100m + n$.

8 Paul and Sara are playing a game with integers on a whiteboard, with Paul going first. When it is Paul's turn, he can pick any two integers on the board and replace them with their product; when it is Sara's turn, she can pick any two integers on the board and replace them with their sum. Play continues until exactly one integer remains on the board. Paul wins if that integer is odd, and Sara wins if it is even.
Initially, there are 2021 integers on the board, each one sampled uniformly at random from the set $\{0, 1, 2, 3, ..., 2021\}$. Assuming both players play optimally, the probability that Paul wins is m/n , where m, n are positive integers and $gcd(m, n) = 1$. Find the remainder when $m + n$ is divided by 1000.

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9 Let N be the smallest positive integer for which $x^2 + x + 1$ divides $166 - \sum_{d|N,d>0} x^d$. Find the remainder when N is divided by 1000.

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10 Three faces X, Y, Z of a unit cube share a common vertex. Suppose the projections of X, Y, Z onto a fixed plane P have areas x, y, z, respectively. If $x : y : z = 6 : 10 : 15$, then $x + y + z$ can be written as m/n , where m, n are positive integers and $gcd(m, n) = 1$. Find $100m + n$.

– General Round

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1 A domino has a left end and a right end, each of a certain color. Alice has four dominos, colored red-red, red-blue, blue-red, and blue-blue. Find the number of ways to arrange the dominos in a row end-to-end such that adjacent ends have the same color. The dominos cannot be rotated.

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2 Suppose a and b are positive integers for which $8a^ab^b = 27a^bb^a$. Find $a^2 + b^2$.

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3 Let $ABCD$ be a unit square. A circle with radius $\frac{32}{49}$ passes through point D and is tangent to side AB at point E. Then $DE = \frac{m}{n}$, where m, n are positive integers and gcd $(m, n) = 1$. Find $100m + n$.

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4 The sum of the digits of the time 19 minutes ago is two less than the sum of the digits of the time right now. Find the sum of the digits of the time in 19 minutes. (Here, we use a standard 12-hour clock of the form $hh : mm$.)

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5 A chord is drawn on a circle by choosing two points uniformly at random along its circumference. This is done two more times to obtain three total random chords. The circle is cut along these three lines, splitting it into pieces. The probability that one of the pieces is a triangle is $\frac{m}{n}$, where m, n are positive integers and gcd $(m, n) = 1$. Find $100m + n$.

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6 Mario has a deck of seven pairs of matching number cards and two pairs of matching Jokers, for a total of 18 cards. He shuffles the deck, then draws the cards from the top one by one until he holds a pair of matching Jokers. The expected number of complete pairs that Mario holds at the end (including the Jokers) is $\frac{m}{n}$, where m, n are positive integers and gcd $(m, n) = 1$. Find $100m + n$.

7 De ne the function $f : R \to R$ by

$$f(x) = \begin{cases} \dfrac{1}{x^2 + \sqrt{x^4 + 2x}} & \text{if } x \notin (-\sqrt[3]{2}, 0] \\ 0 \ , & \text{otherwise} \end{cases}$$

The sum of all real numbers x for which $f^{10}(x) = 1$ can be written as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers, d is positive, c is square-free, and gcd$(a, b, d) = 1$. Find $1000a + 100b + 10c + d$.
(Here, $f^n(x)$ is the function $f(x)$ iterated n times. For example, $f^3(x) = f(f(f(x)))$.)

8 Eight points are chosen on the circumference of a circle, labelled P_1, P_2, ..., P_8 in clockwise order. A route is a sequence of at least two points P_{a_1}, P_{a_2}, ..., P_{a_n} such that if an ant were to visit these points in their given order, starting at P_{a_1} and ending at P_{a_n}, by following $n-1$ straight line segments (each connecting each P_{a_i} and $P_{a_{i+1}}$), it would never visit a point twice or cross its own path. Find the number of routes.

9 $ABCDE$ is a cyclic convex pentagon, and $AC = BD = CE$. AC and BD intersect at X, and BD and CE intersect at Y . If $AX = 6$, $XY = 4$, and $YE = 7$, then the area of pentagon $ABCDE$ can be written as $\frac{a\sqrt{b}}{c}$, where a, b, c are integers, c is positive, b is square-free, and gcd$(a, c) = 1$. Find $100a + 10b + c$.

10 Real numbers x, y, z satisfy

$$x + xy + xyz = 1, y + yz + xyz = 2, z + xz + xyz = 4.$$

The largest possible value of xyz is $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers, d is positive, c is square-free, and gcd$(a, b, d) = 1$. Find $1000a + 100b + 10c + d$.

– Theme Round

1 Let n be the answer to this problem. In acute triangle ABC, point D is located on side BC so that $\angle BAD = \angle DAC$ and point E is located on AC so that $BE \perp AC$. Segments BE and AD intersect at X such that $\angle BXD = n^o$: Given that $\angle XBA = 16^o$, find the measure of $\angle BCA$.

2 Let n be the answer to this problem. An urn contains white and black balls. There are n white balls and at least two balls of each color in the urn. Two balls are randomly drawn from the urn without replacement. Find the probability, in percent, that the rst ball drawn is white and the second is black.

3 Let n be the answer to this problem. Hexagon $ABCDEF$ is inscribed in a circle of radius 90. The area of $ABCDEF$ is $8n$, $AB = BC = DE = EF$, and $CD = FA$. Find the area of triangle ABC:

4 Let n be the answer to this problem. We de fine the digit sum of a date as the sum of its 4 digits when expressed in mmdd format (e.g. the digit sum of 13 May is $0+5+1+3=9$). Find the number of dates in the year 2021 with digit sum equal to the positive integer n.

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5 Let n be the answer to this problem. The polynomial $x^n + ax^2 + bx + c$ has real coefficients and exactly k real roots. Find the sum of the possible values of k.

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6 Let n be the answer to this problem. a and b are positive integers satisfying

$$3a+5b \equiv 19 \pmod{n+1}$$

$$4a+2b \equiv 25 \pmod{n+1}$$

Find $2a+6b$.

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7 Let n be the answer to this problem. Box B initially contains n balls, and Box A contains half as many balls as Box B. After 80 balls are moved from Box A to Box B, the ratio of balls in Box A to Box B is now $\frac{p}{q}$, where p, q are positive integers with gcd$(p,q)=1$. Find $100p+q$.

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8 Let n be the answer to this problem. Find the number of distinct (i.e. non-congruent), non-degenerate triangles with integer side lengths and perimeter n.

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9 Let n be the answer to this problem. Find the minimum number of colors needed to color the divisors of $(n-24)!$ such that no two distinct divisors s, t of the same color satisfy $s|t$.

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10 Let n be the answer to this problem. Suppose square $ABCD$ has side-length 3. Then, congruent non-overlapping squares $EHGF$ and $IHJK$ of side-length $\frac{n}{6}$ are drawn such that A,C, and H are collinear, E lies on BC and I lies on CD. Given that AJG is an equilateral triangle, then the area of AJG is $a+b\sqrt{c}$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a+b+c$.
