Stanford Mathematics Tournament 2019

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– Team Round

– **p1.** Given $x + y = 7$, find the value of x that minimizes $4x^2 + 12xy + 9y^2$.

p2. There are real numbers b and c such that the only x-intercept of $8y = x^2 + bx + c$ equals its y-intercept. Compute $b + c$.

p3. Consider the set of 5 digit numbers $ABCDE$ (with $A \ne 0$) such that $A+B = C$, $B+C = D$, and $C + D = E$. What's the size of this set?

p4. Let D be the midpoint of BC in $\triangle ABC$. A line perpendicular to D intersects AB at E. If the area of $\triangle ABC$ is four times that of the area of $\triangle BDE$, what is $\angle ACB$ in degrees?

p5. Define the sequence $c_0, c_1, ...$ with $c_0 = 2$ and $c_k = 8c_{k-1} + 5$ for $k > 0$. Find $\lim_{k\to\infty} \frac{c_k}{8^k}$.

p6. Find the maximum possible value of $|\sqrt{n^2 + 4n + 5} - \sqrt{n^2 + 2n + 5}|$.

p7. Let $f(x) = \sin^8(x) + \cos^8(x) + \frac{3}{8}\sin^4(2x)$. Let $f^{(n)}$ (x) be the nth derivative of f. What is the largest integer a such that 2^a divides $f^{(2020)}(15^o)$?

p8. Let R^n be the set of vectors $(x_1, x_2, ..., x_n)$ where $x_1, x_2, ..., x_n$ are all real numbers. Let $||(x_1, ..., x_n)||$ denote $\sqrt{x_1^2 + ... + x_n^2}$. Let S be the set in R^9 given by

$$S = \{(x, y, z) : x, y, z \in R^3, 1 = ||x|| = ||y - x|| = ||z - y||\}.$$

If a point (x, y, z) is uniformly at random from S, what is $E[||z||^2]$?

p9. Let $f(x)$ be the unique integer between 0 and $x - 1$, inclusive, that is equivalent modulo x to $\left(\sum_{i=0}^{2} \binom{x-1}{i}((x-1-i)! + i!)\right)$. Let S be the set of primes between 3 and 30, inclusive. Find $\sum_{x\in S}^{f(x)}$.

p10. In the Cartesian plane, consider a box with vertices $(0, 0)$,$(\frac{22}{7}, 0)$,$(0, 24)$,$(\frac{22}{7}, 4)$. We pick an integer a between 1 and 24, inclusive, uniformly at random. We shoot a puck from $(0, 0)$ in the direction of $(\frac{22}{7}, a)$ and the puck bounces perfectly around the box (angle in equals angle out, no friction) until it hits one of the four vertices of the box. What is the expected number of times it will hit an edge or vertex of the box, including both when it starts at $(0, 0)$ and when it ends at some vertex of the box?

p11. Sarah is buying school supplies and she has $\$2019$. She can only buy full packs of each of the following items. A pack of pens is $\$4$, a pack of pencils is $\$3$, and any type of notebook or stapler is $\$1$. Sarah buys at least 1 pack of pencils. She will either buy 1 stapler or no stapler. She will buy at most 3 college-ruled notebooks and at most 2 graph paper notebooks. How many ways can she buy school supplies?

p12. Let O be the center of the circumcircle of right triangle ABC with $\angle ACB = 90^o$. Let M be the midpoint of minor arc AC and let N be a point on line BC such that $MN \perp BC$. Let P be the intersection of line AN and the Circle O and let Q be the intersection of line BP and MN. If $QN = 2$ and $BN = 8$, compute the radius of the Circle O.

p13. Reduce the following expression to a simplified rational

$$\frac{1}{1-\cos\frac{\pi}{9}} + \frac{1}{1-\cos\frac{5\pi}{9}} + \frac{1}{1-\cos\frac{7\pi}{9}}$$

p14. Compute the following integral $\int_0^{\infty} \log(1 + e^{-t})dt$.

p15. Define $f(n)$ to be the maximum possible least-common-multiple of any sequence of positive integers which sum to n. Find the sum of all possible odd $f(n)$

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Geometry Round

1 Let $ABCD$ be a unit square. A semicircle with diameter AB is drawn so that it lies outside of the square. If E is the midpoint of arc AB of the semicircle, what is the area of triangle CDE

2 A cat and mouse live on a house mapped out by the points $(-1, 0)$, $(-1, 2)$, $(0, 3)$, $(1, 2)$, $(1, 0)$. The cat starts at the top of the house (point $(0, 3)$) and the mouse starts at the origin (0, 0). Both start running clockwise around the house at the same time. If the cat runs at 12 units a minute and the mouse at 9 units a minute, how many laps around the house will the cat run before it catches the mouse?

3 In triangle ABC with $AB = 10$, let$D$ be a point on side BC such that AD bisects $\angle BAC$. If $\frac{CD}{BD} = 2$ and the area of ABC is 50, compute the value of $\angle BAD$ in degrees.

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4 Let ω_1 and ω_2 be two circles intersecting at points P and Q. The tangent line closer to Q touches ω_1 and ω_2 at M and N respectively. If $PQ = 3$, $QN = 2$, and $MN = PN$, what is QM^2?

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5 The bases of a right hexagonal prism are regular hexagons of side length $s > 0$, and the prism has height h. The prism contains some water, and when it is placed on a flat surface with a hexagonal face on the bottom, the water has depth $\frac{s\sqrt{3}}{4}$. The water depth doesn't change when the prism is turned so that a rectangular face is on the bottom. Compute $\frac{h}{s}$.

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6 Let the altitude of $\triangle ABC$ from A intersect the circumcircle of $\triangle ABC$ at D. Let E be a point on line AD such that $E \neq A$ and $AD = DE$. If $AB = 13$, $BC = 14$, and $AC = 15$, what is the area of quadrilateral $BDCE$?

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7 Let G be the centroid of triangle ABC with $AB = 9$, $BC = 10$, and $AC = 17$. Denote D as the midpoint of BC. A line through G parallel to BC intersects AB at M and AC at N. If BG intersects CM at E and CG intersects BN at F, compute the area of triangle DEF.

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8 In the coordinate plane, a point A is chosen on the line $y = \frac{3}{2}x$ in the first quadrant. Two perpendicular lines ℓ_1 and ℓ_2 intersect at A where ℓ_1 has slope $m > 1$. Let ℓ_1 intersect the x-axis at B, and ℓ_2 intersects the x and y axes at C and D, respectively. Suppose that line BD has slope $-m$ and $BD = 2$. Compute the length of CD.

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9 Let $ABCD$ be a quadrilateral with $\angle ABC = \angle CDA = 45^o$, $AB = 7$, and $BD = 25$. If AC is perpendicular to CD, compute the length of BC.

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10 Let ABC be an acute triangle with $BC = 48$. Let M be the midpoint of BC, and let D and E be the feet of the altitudes drawn from B and C to AC and AB respectively. Let P be the intersection between the line through A parallel to BC and line DE. If $AP = 10$, compute the length of PM,

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– Geometry Tiebreaker

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1 Let $ABCD$ be a quadrilateral with $\angle DAB = \angle ABC = 120^o$. If $AB = 3$, $BC = 2$, and $AD = 4$, what is the length of CD?

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2 Let $ABCD$ be a rectangle with $AB = 8$ and $BC = 6$. Point E is outside of the rectangle such that $CE = DE$. Point D is reflected over line AE so that its image, D' , lies on the interior of the rectangle. Point D' is then reflected over diagonal AC, and its image lies on side AB. What is the length of DE?

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3 Right triangle ABC with $\angle ABC = 90^o$ is inscribed in a circle ω_1 with radius 3. A circle ω_2 tangent to AB, BC, and ω_1 has radius 2. Compute the area of $\triangle ABC$.