Problems from the first round of Kyiv Mathematical Olympiad 2022

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by MS_Kekas

– Grade 7

Problem 1 Represent $\frac{1}{2021}$ as a difference of two irreducible fractions with smaller denominators.

(Proposed by Bogdan Rublov)

Problem 2 There are n sticks which have distinct integer length. Suppose that it's possible to form a non-degenerate triangle from any 3 distinct sticks among them. It's also known that there are sticks of lengths 5 and 12 among them. What's the largest possible value of n under such conditions?

(Proposed by Bogdan Rublov)

Problem 3 You are given n not necessarily distinct real numbers $a_1, a_2, \ldots, a_n$. Let's consider all $2^n - 1$ ways to select some nonempty subset of these numbers, and for each such subset calculate the sum of the selected numbers. What largest possible number of them could have been equal to 1?

For example, if $a = [-1, 2, 2]$, then we got 3 once, 4 once, 2 twice, -1 once, 1 twice, so the total number of ones here is 2.

(Proposed by Anton Trygub)

Problem 4 In some magic country, there are banknotes only of values 3, 25, 80 hryvnyas. Businessman Victor ate in one restaurant of this country for 2024 days in a row, and each day (except the first) he spent exactly 1 hryvnya more than the day before (without any change). Could he have spent exactly 1000000 banknotes?

(Proposed by Oleksii Masalitin)

– Grade 8

Problem 1 Consider 5 distinct positive integers. Can their mean be

a)Exactly 3 times larger than their largest common divisor?

b)Exactly 2 times larger than their largest common divisor?

Problem 2 Same as 7.3

Problem 3 In triangle ABC $\angle B > 90^\circ$. Tangents to this circle in points A and B meet at point P, and the line passing through B perpendicular to BC meets the line AC at point K. Prove that $PA = PK$.

(Proposed by Danylo Khilko)

Problem 4 What's the largest number of integers from 1 to 2022 that you can choose so that no sum of any two different chosen integers is divisible by any difference of two different chosen integers?

(Proposed by Oleksii Masalitin)

Problem 5 2022 teams participated in an underwater polo tournament, each two teams played exactly once against each other. Team receives $2, 1, 0$ points for win, draw, and loss correspondingly. It turned out that all teams got distinct numbers of points. In the final standings the teams were ordered by the total number of points.

A few days later, organizers realized that the results in the final standings were wrong due to technical issues: in fact, each match that ended with a draw according to them in fact had a winner, and each match with a winner in fact ended with a draw. It turned out that all teams still had distinct number of points! They corrected the standings, and ordered them by the total number of points.

Could the correct order turn out to be the reversed initial order?

(Proposed by Fedir Yudin)

– Grade 9

Problem 1 What's the smallest possible value of

$$\frac{(x+y+|x-y|)^2}{xy}$$

over positive real numbers x, y?

Problem 2 For any reals x, y, show the following inequality:

$$\sqrt{(x+4)^2+(y+2)^2}+\sqrt{(x-5)^2+(y+4)^2} \leq \sqrt{(x-2)^2+(y-6)^2}+\sqrt{(x-5)^2+(y-6)^2}+20$$

(Proposed by Bogdan Rublov)

Problem 3 Let AL be the inner bisector of triangle ABC. The circle centered at B with radius BL meets the ray AL at points L and E, and the circle centered at C with radius CL meets the ray AL at points L and D. Show that $AL^2 = AE \times AD$.

(Proposed by Mykola Moroz)

Problem 4 Let's call integer square-free if it's not divisible by p^2 for any prime p. You are given a square-free integer $n > 1$, which has exactly d positive divisors. Find the largest number of its divisors that you can choose, such that $a^2 + ab - n$ isn't a square of an integer for any a, b among chosen divisors.

(Proposed by Oleksii Masalitin)

Problem 5 $n \geq 2$ teams participated in an underwater polo tournament, each two teams played exactly once against each other. A team receives $2, 1, 0$ points for a win, draw, and loss correspondingly. It turned out that all teams got distinct numbers of points. In the final standings, the teams were ordered by the total number of points.

A few days later, organizers realized that the results in the final standings were wrong due to technical issues: in fact, each match that ended with a draw according to them in fact had a winner, and each match with a winner in fact ended with a draw. It turned out that all teams still had distinct number of points! They corrected the standings and ordered them by the total number of points.

For which n could the correct order turn out to be the reversed initial order?

(Proposed by Fedir Yudin)

– Grade 10

Problem 1 Does there exist a quadratic trinomial $ax^2 + bx + c$ such that a, b, c are odd integers, and $\frac{1}{2022}$ is one of its roots?

Problem 2 You are given $2n$ distinct integers. What's the largest integer C such that you can always form at least C pairs from them, so that no integer is in more than one pair, and the sum of integers in each pair is a composite number?

(Proposed by Anton Trygub)

Problem 3 Diagonals of a cyclic quadrilateral $ABCD$ intersect at point P. The circumscribed circles of triangles APD and BPC intersect the line AB at points E, F correspondingly. Q and R are the projections of P onto the lines FC, DE correspondingly. Show that $AB \parallel QR$.

(Proposed by Mykhailo Shtandenko)

Problem 4 For any nonnegative reals x, y show the inequality

$$x^2y^2 + x^2y + xy^2 \leq x^4y + x + y^4$$

.

Problem 5 There is a black token in the lower-left corner of a board $m \times n$ ($m, n \geq 3$), and there are white tokens in the lower-right and upper-left corners of this board. Petryk and Vasyl are playing a game, with Petryk playing with a black token and Vasyl with white tokens. Petryk moves first.

In his move, a player can perform the following operation at most two times: choose any his token and move it to any adjacent by side cell, with one restriction: you can't move a token to a cell where at some point was one of the opponents' tokens.

Vasyl wins if at some point of the game white tokens are in the same cell. For which values of m, n can Petryk prevent him from winning?

(Proposed by Arsenii Nikolaiev)

– Grade 11

Problem 1 The teacher wrote 5 distinct real numbers on the board. After this, Petryk calculated the sums of each pair of these numbers and wrote them on the left part of the board, and Vasyl calculated the sums of each triple of these numbers and wrote them on the left part of the board (each of them wrote 10 numbers). Could the multisets of numbers written by Petryk and Vasyl be identical?

Problem 2 Same as 10.2

Problem 3 Let H and O be the orthocenter and the circumcenter of the triangle ABC. Line OH intersects the sides AB, AC at points X, Y correspondingly, so that H belongs to the segment OX. It turned out that $XH = HO = OY$. Find $\angle BAC$.

(Proposed by Oleksii Masalitin)

Problem 4 You are given $n \geq 4$ positive real numbers. It turned out that all $\frac{n(n-1)}{2}$ of their pairwise products form an arithmetic progression in some order. Show that all given numbers are equal.

(Proposed by Anton Trygub)

Problem 5 Find the smallest integer n for which it's possible to cut a square into $2n$ squares of two sizes: n squares of one size, and n squares of another size.

(Proposed by Bogdan Rublov)