Harvard-MIT November Tournament, Harvard-MIT Mathematics Tournament November 2014

www.artofproblemsolving.com/community/c2883099

by parmenides51, I_m_vanu1996, Kingofmath101, shootingstar8

–	Team Round
1	What is the smallest positive integer n which cannot be written in any of the following forms? • $n = 1 + 2 + ... + k$ for a positive integer k. • $n = p^k$ for a prime number p and integer k. • $n = p + 1$ for a prime number p.
2	Let $f(x) = x^2 + 6x + 7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x.
3	The side lengths of a triangle are distinct positive integers. One of the side lengths is a multiple of 42, and another is a multiple of 72. What is the minimum possible length of the third side?
4	How many ways are there to color the vertices of a triangle red, green, blue, or yellow such that no two vertices have the same color? Rotations and reflections are considered distinct.
5	Let A, B, C, D, E be five points on a circle; some segments are drawn between the points so that each of the $5C2 = 10$ pairs of points is connected by either zero or one segments. Determine the number of sets of segments that can be drawn such that: • It is possible to travel from any of the five points to any other of the five points along drawn segments. • It is possible to divide the five points into two nonempty sets S and T such that each segment has one endpoint in S and the other endpoint in T.
6	Find the number of strictly increasing sequences of nonnegative integers with the following properties: • The first term is 0 and the last term is 12. In particular, the sequence has at least two terms. • Among any two consecutive terms, exactly one of them is even.

7	Sammy has a wooden board, shaped as a rectangle with length 2^{2014} and height 3^{2014}. The board is divided into a grid of unit squares. A termite starts at either the left or bottom edge of the rectangle, and walks along the gridlines by moving either to the right or upwards, until it reaches an edge opposite the one from which the termite started. Depicted below are two possible paths of the termite.
	`https://cdn.artofproblemsolving.com/attachments/3/0/39f3b2aa9c61ff24ffc22b968790f4c61da6f`
	`png`
	The termite's path dissects the board into two parts. Sammy is surprised to find that he can still arrange the pieces to form a new rectangle not congruent to the original rectangle. This

rectangle has perimeter P. How many possible values of P are there?

8 Let H be a regular hexagon with side length one. Peter picks a point P uniformly and at random within H, then draws the largest circle with center P that is contained in H. What is this probability that the radius of this circle is less than $1/2$?

9 How many lines pass through exactly two points in the following hexagonal grid?
`https://cdn.artofproblemsolving.com/attachments/2/e/35741c80d0e0ee0ca56f1297b1e377c8db9e2`
`png`

10 Let $ABCDEF$ be a convex hexagon with the following properties.
(a) $\overline{AC}$ and $\overline{AE}$ trisect $\angle BAF$.
(b) $\overline{BE} \parallel \overline{CD}$ and $\overline{CF} \parallel \overline{DE}$.
(c) $AB = 2AC = 4AE = 8AF$.
Suppose that quadrilaterals $ACDE$ and $ADEF$ have area 2014 and 1400, respectively. Find the area of quadrilateral $ABCD$.

– General Round

1 Two circles ω and γ have radii 3 and 4 respectively, and their centers are 10 units apart. Let x be the shortest possible distance between a point on ω and a point on γ , and let$y$ be the longest possible distance between a point on ω and a point on γ . Find the product xy.

2 Let ABC be a triangle with $\angle B = 90^o$. Given that there exists a point D on AC such that $AD = DC$ and $BD = BC$, compute the value of the ratio $\frac{AB}{BC}$.

3 Compute the greatest common divisor of $4^8 - 1$ and $8^{12} - 1$.

4 In rectangle $ABCD$ with area 1, point M is selected on $\overline{AB}$ and points X, Y are selected on $\overline{CD}$ such that $AX < AY$. Suppose that $AM = BM$. Given that the area of triangle MXY is $\frac{1}{2014}$, compute the area of trapezoid $AXYB$.

5 Mark and William are playing a game with a stored value. On his turn, a player may either multiply the stored value by 2 and add 1 or he may multiply the stored value by 4 and add 3. The first player to make the stored value exceed 2^{100} wins. The stored value starts at 1 and Mark goes first. Assuming both players play optimally, what is the maximum number of times that William can make a move?

(By optimal play, we mean that on any turn the player selects the move which leads to the best possible outcome given that the opponent is also playing optimally. If both moves lead to the same outcome, the player selects one of them arbitrarily.)

6 Let ABC be a triangle with $AB = 5$, $AC = 4$, $BC = 6$. The angle bisector of C intersects side AB at X. Points M and N are drawn on sides BC and AC, respectively, such that $\overline{XM} \parallel \overline{AC}$ and $\overline{XN} \parallel \overline{BC}$. Compute the length MN.

---

7 Consider the set of 5-tuples of positive integers at most 5. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is *perfect* if for any distinct indices i, j, k, the three numbers a_i, a_j , a_k do not form an arithmetic progression (in any order). Find the number of perfect 5-tuples.

---

8 Let a, b, c, x be reals with $(a + b)(b + c)(c + a) \neq 0$ that satisfy

$$\frac{a^2}{a+b} = \frac{a^2}{a+c} + 20, \quad \frac{b^2}{b+c} = \frac{b^2}{b+a} + 14, \text{and } \frac{c^2}{c+a} = \frac{c^2}{c+b} + x.$$

Compute x.

---

9 For any positive integers a and b, define $a \oplus b$ to be the result when adding a to b in binary (base 2), neglecting any carry-overs. For example, $20 \oplus 14 = 10100_2 \oplus 1110_2 = 11010_2 = 26$.
(The operation $\oplus$ is called the *exclusive or.*)
Compute the sum

$$\sum_{k=0}^{2^{2014}-1} \left(k \oplus \left\lfloor \frac{k}{2} \right\rfloor \right).$$

Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.

---

10 Suppose that m and n are integers with $1 \le m \le 49$ and $n \ge 0$ such that m divides $n^{n+1} + 1$. What is the number of possible values of m?

– Theme Round

1-5 Townspeople and Goons

In the city of Lincoln, there is an empty jail, at least two townspeople and at least one goon. A game
proceeds over several days, starting with morning.

• Each morning, one randomly selected unjailed person is placed in jail. If at this point all goons are jailed, and at least one townsperson remains, then the townspeople win. If at this point all townspeople are jailed and at least one goon remains, then the goons win. • Each evening, if there is at least one goon and at least one townsperson not in jail, then one randomly selected townsperson is jailed. If at this point there are at least as many goons remaining as townspeople remaining, then the goons win.

The game ends immediately after any group wins.

p1. Find the probability that the townspeople win if there are initially two townspeople and one goon.

p2. Find the smallest positive integer n such that, if there are initially $2n$ townspeople and 1 goon, then the probability the townspeople win is greater than 50%.

p3. Find the smallest positive integer n such that, if there are initially $n+1$ townspeople and n goons, then the probability the townspeople win is less than 1%.

p4. Suppose there are initially 1001 townspeople and two goons. What is the probability that, when the game ends, there are exactly 1000 people in jail?

p5. Suppose that there are initially eight townspeople and one goon. One of the eight townspeople is named Jester. If Jester is sent to jail during some morning, then the game ends immediately in his sole victory. (However, the Jester does not win if he is sent to jail during some night.)
Find the probability that only the Jester wins.

---

6 Let P_1, P_2, P_3 be pairwise distinct parabolas in the plane. Find the maximum possible number of intersections between two or more of the P_i. In other words, find the maximum number of points that can lie on two or more of the parabolas P_1, P_2, P_3 .

---

7 Let P be a parabola with focus F and directrix ℓ. A line through F intersects P at two points A and B. Let D and C be the feet of the altitudes from A and B onto ℓ, respectively. Given that $AB = 20$ and $CD = 14$, compute the area of $ABCD$.

---

8 Consider the parabola consisting of the points (x, y) in the real plane satisfying

$$(y + x) = (y - x)^2 + 3(y - x) + 3.$$

Find the minimum possible value of y.

---

9 In equilateral triangle ABC with side length 2, let the parabola with focus A and directrix BC intersect sides AB and AC at A_1 and A_2, respectively. Similarly, let the parabola with focus B and directrix CA intersect sides BC and BA at B_1 and B_2, respectively. Finally, let the parabola with focus C and directrix AB intersect sides CA and C_B at C_1 and C_2, respectively.
Find the perimeter of the triangle formed by lines A_1A_2, B_1B_2, C_1C_2.

---

10 Let z be a complex number and k a positive integer such that z^k is a positive real number other than 1. Let $f(n)$ denote the real part of the complex number z^n. Assume the parabola $p(n) = an^2 + bn + c$ intersects $f(n)$ four times, at $n = 0, 1, 2, 3$. Assuming the smallest possible value of k, find the largest possible value of a.
