Romania Team Selection Test 2017

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–	Day 1
P1	Let $ABCD$ be a trapezium, $AD \parallel BC$, and let E, F be points on the sidesAB and CD, respectively. The circumcircle of AEF meets AD again at A_1, and the circumcircle of CEF meets BC again at C_1. Prove that A_1C_1, BD, EF are concurrent.
P2	Consider a finite collection of 3-element sets A_i, no two of which share more than one element, whose union has cardinality 2017. Show that the elements of this union can be coloured with two colors, blue and red, so that at least 64 elements are blue and each A_i has at least one red element.
P3	Consider the sequence of rational numbers defined by $x_1 = \frac{4}{3}$, and $x_{n+1} = \frac{x_n^2}{x_n^2 - x_n + 1}$. Show that the nu,erator of the lowest term expression of each sum $x_1 + x_2 + ... + x_k$ is a perfect square.
P4	Determine the smallest radius a circle passing through EXACTLY three lattice points may have.
P5	A planar country has an odd number of cities separated by pairwise distinct distances. Some of these cities are connected by direct two-way flights. Each city is directly connected to exactly two ther cities, and the latter are located farthest from it. Prove that, using these flights, one may go from any city to any other city
–	Day 2
P1	Let ABC be a triangle with $AB < AC$, let G, H be its centroid and otrhocenter. Let D be the otrhogonal projection of A on the line BC, and let M be the midpoint of the side BC. The circumcircle of ABC crosses the ray HM emanating from M at P and the ray DG emanating from D at Q, outside the segment DG. Show that the lines DP and MQ meet on the circumcircle of ABC.
P2	Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints: -each cell contains a distinct divisor; -the sums of all rows are equal; and -the sums of all columns are equal.
P3	Given an interger $n \geq 2$, determine the maximum value the sum $\frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_{n-1}}{a_n}$ may achieve, and the points at which the maximum is achieved, as $a_1, a_2, ...a_n$ run over all positive real numers

subject to $a_k \geq a_1 + a_2... + a_{k-1}$, for $k = 2, ...n$

P4 Given a positive odd integer n, show that the arithmetic mean of fractional parts $\{\frac{k^{2n}}{p}\}, k = 1, ..., \frac{p-1}{2}$ is the same for infinitely many primes p .

– Day 3

P1 a) Determine all 4-tuples (x_0, x_1, x_2, x_3) of pairwise distinct intergers such that each x_k is coprime to x_{k+1}(indices reduces modulo 4) and the cyclic sum $\frac{x_0}{x_1} + \frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_1}$ is an interger.
b)Show that there are infinitely many 5-tuples $(x_0, x_1, x_2, x_3, x_4)$ of pairwise distinct intergers such that each x_k is coprime to x_{k+1}(indices reduces modulo 5) and the cyclic sum $\frac{x_0}{x_1} + \frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_4} + \frac{x_4}{x_0}$ is an interger.

P2 Determine all intergers $n \geq 2$ such that $a + \sqrt{2}$ and $a^n + \sqrt{2}$ are both rational for some real number a depending on n

P3 Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

P4 Let $ABCD$ be a convex quadrilateral and let P and Q be variable points inside this quadrilateral so that $\angle APB = \angle CPD = \angle AQB = \angle CQD$. Prove that the lines PQ obtained in this way all pass through a fixed point , or they are all parallel.

– Day 4

P1 Let m be a positive interger, let p be a prime, let $a_1 = 8p^m$, and let $a_n = (n+1)^{\frac{a_{n-1}}{n}}$, $n = 2, 3....$ Determine the primes p for which the products $a_n(1 - \frac{1}{a_1})(1 - \frac{1}{a_2})...(1 - \frac{1}{a_n})$, $n = 1, 2, 3...$ are all integral.

P2 Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers a_1, a_2, a_3, a_4, a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k, l such that

$$\left| \frac{a_i}{a_j} - \frac{a_k}{a_l} \right| \leq C.$$

P3 Let $n \geq 3$ be a positive integer. Find the maximum number of diagonals in a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

– Day 5

P1 Consider fractions $\frac{a}{b}$ where a and b are positive integers.
(a) Prove that for every positive integer n, there exists such a fraction $\frac{a}{b}$ such that $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$ and $b \leq \sqrt{n}+1$.
(b) Show that there are infinitely many positive integers n such that no such fraction $\frac{a}{b}$ satisfies $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$ and $b \leq \sqrt{n}$.

P2 Let n be a positive integer, and let S_n be the set of all permutations of $1,2,...,n$. let k be a non-negative integer, let $a_{n,k}$ be the number of even permutations σ in S_n such that $\sum_{i=1}^{n}|\sigma(i)-i|=2k$ and $b_{n,k}$ be the number of odd permutations σ in S_n such that $\sum_{i=1}^{n}|\sigma(i)-i|=2k$. Evaluate $a_{n,k}-b_{n,k}$.

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P3 Let $ABCD$ be a convex quadrilateral with $\angle ABC=\angle ADC<90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ\perp AC$.
