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by v_Enhance

Day 1 June 25, 2016

1 Let $A = A(x, y)$ and $B = B(x, y)$ be two-variable polynomials with real coefficients. Suppose that $A(x, y)/B(x, y)$ is a polynomial in x for infinitely many values of y, and a polynomial in y for infinitely many values of x. Prove that B divides A, meaning there exists a third polynomial C with real coefficients such that $A = B \cdot C$.

Proposed by Victor Wang

2 Let ABC be a scalene triangle with orthocenter H and circumcenter O. Denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line AN at a point $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect at a point T on $\overline{PN}$.

Proposed by Evan Chen

3 Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \dots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo n.

Proposed by Yang Liu

Day 2 June 27, 2016

4 Suppose that n and k are positive integers such that

$$1 = \underbrace{\varphi(\varphi(\dots\varphi(}_{k \text{ times}}n)\dots)).$$

Prove that $n \leq 3^k$.

Here $\varphi(n)$ denotes Euler's totient function, i.e. $\varphi(n)$ denotes the number of elements of $\{1, \dots, n\}$ which are relatively prime to n. In particular, $\varphi(1) = 1$.

Proposed by Linus Hamilton

5 In the coordinate plane are finitely many *walls*; which are disjoint line segments, none of which are parallel to either axis. A bulldozer starts at an arbitrary point and moves in the $+x$ direction. Every time it hits a wall, it turns at a right angle to its path, away from the wall, and continues moving. (Thus the bulldozer always moves parallel to the axes.)

Prove that it is impossible for the bulldozer to hit both sides of every wall.

Proposed by Linus Hamilton and David Stoner

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6 Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.

Proposed by Danielle Wang
