Problems from the second round of Kyiv Mathematical Olympiad 2022

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by MS_Kekas

– Grade 7

Problem 1 a) Do there exist positive integers a and d such that $[a, a+d] = [a, a+2d]$?

b) Do there exist positive integers a and d such that $[a, a+d] = [a, a+4d]$?

Here $[a, b]$ denotes the least common multiple of integers a, b.

Problem 2 There is a central train station in point O, which is connected to other train stations $A_1, A_2, \dots, A_8$ with tracks. There is also a track between stations A_i and A_{i+1} for each i from 1 to 8 (here $A_9 = A_1$). The length of each track A_iA_{i+1} is equal to 1, and the length of each track OA_i is equal to 2, for each i from 1 to 8.

There are also 8 trains $B_1, B_2, \dots, B_8$, with speeds $1, 2, \dots, 8$ correspondently. Trains can move only by the tracks above, in both directions. No time is wasted on changing directions. If two or more trains meet at some point, they will move together from now on, with the speed equal to that of the fastest of them.

Is it possible to arrange trains into stations $A_1, A_2, \dots, A_8$ (each station has to contain one train initially), and to organize their movement in such a way, that all trains arrive at O in time $t < \frac{1}{2}$?

(Proposed by Bogdan Rublov)

Problem 3 In triangle ABC the median BM is equal to half of the side BC. Show that $\angle ABM = \angle BCA + \angle BAC$.

(Proposed by Anton Trygub)

Problem 4 Fedir and Mykhailo have three piles of stones: the first contains 100 stones, the second 101, the third 102. They are playing a game, going in turns, Fedir makes the first move. In one move player can select any two piles of stones, let's say they have a and b stones left correspondently, and remove $gcd(a, b)$ stones from each of them. The player after whose move some pile becomes empty for the first time wins. Who has a winning strategy?

As a reminder, $gcd(a, b)$ denotes the greatest common divisor of a, b.

(Proposed by Oleksii Masalitin)

– Grade 8

Problem 1 Find all triples (a, b, c) of positive integers for which $a + [a, b] = b + [b, c] = c + [c, a]$.

Here $[a, b]$ denotes the least common multiple of integers a, b.

(Proposed by Mykhailo Shtandenko)

Problem 2 Monica and Bogdan are playing a game, depending on given integers n, k. First, Monica writes some k positive numbers. Bogdan wins, if he is able to find n points on the plane with the following property: for any number m written by Monica, there are some two points chosen by Bogdan with distance exactly m between them. Otherwise, Monica wins.

Determine who has a winning strategy depending on n, k.

(Proposed by Fedir Yudin)

Problem 3 Nonzero real numbers $x_1, x_2, \ldots, x_n$ satisfy the following condition:

$$x_1 - \frac{1}{x_2} = x_2 - \frac{1}{x_3} = \ldots = x_{n-1} - \frac{1}{x_n} = x_n - \frac{1}{x_1}$$

Determine all n for which $x_1, x_2, \ldots, x_n$ have to be equal.

(Proposed by Oleksii Masalitin, Anton Trygub)

Problem 4 Points D, E, F are selected on sides BC, CA, AB correspondingly of triangle ABC with $\angle C = 90^\circ$ such that $\angle DAB = \angle CBE$ and $\angle BEC = \angle AEF$. Show that $DB = DF$.

(Proposed by Mykhailo Shtandenko)

– Grade 9

Problem 1 Find all triples (a, b, c) of positive integers for which $a + (a, b) = b + (b, c) = c + (c, a)$.

Here (a, b) denotes the greatest common divisor of integers a, b.

(Proposed by Mykhailo Shtandenko)

Problem 2 2022 points are arranged in a circle, one of which is colored in black, and others in white. In one operation, The Hedgehog can do one of the following actions:

1) Choose two adjacent points of the same color and flip the color of both of them (white becomes black, black becomes white)

2) Choose two points of opposite colors with exactly one point in between them, and flip the color of both of them

Is it possible to achieve a configuration where each point has a color opposite to its initial color with these operations?

(Proposed by Oleksii Masalitin)

Problem 3 Same as 8.3

Problem 4 Let ω denote the circumscribed circle of triangle ABC, I be its incenter, and K be any point on arc AC of ω not containing B. Point P is symmetric to I with respect to point K. Point T on arc AC of ω containing point B is such that $\angle KCT = \angle PCI$. Show that the bisectors of angles AKC and ATC meet on line CI.

(Proposed by Anton Trygub)

– Grade 10

Problem 1 Positive reals x, y, z satisfy

$$\frac{xy+1}{x+1} = \frac{yz+1}{y+1} = \frac{zx+1}{z+1}$$

Do they all have to be equal?

(Proposed by Oleksii Masalitin)

Problem 2 Same as 9.2

Problem 3 Let AH_A, BH_B, CH_C be the altitudes of triangle ABC. Prove that if $\frac{H_BC}{AC} = \frac{H_CA}{AB}$, then the line symmetric to BC with respect to line H_BH_C is tangent to the circumscribed circle of triangle H_BH_CA.

(Proposed by Mykhailo Bondarenko)

Problem 4 Prime $p > 2$ and a polynomial Q with integer coefficients are such that there are no integers $1 \le i < j \le p-1$ for which $(Q(j) - Q(i))(jQ(j) - iQ(i))$ is divisible by p. What is the smallest possible degree of Q?

(Proposed by Anton Trygub)

– Grade 11

Problem 1 Same as 8.2

Problem 2 Initially memory of computer contained a single polynomial $x^2 - 1$. Every minute computer chooses any polynomial $f(x)$ from its memory and writes $f(x^2-1)$ and $f(x)^2-1$ to it, or chooses any two distinct polynomials $g(x), h(x)$ from its memory and writes polynomial $\frac{g(x)+h(x)}{2}$ to it (no polynomial is ever erased from its memory). Can it happen that after some time, memory of computer contains $P(x) = \frac{1}{1024}(x^2-1)^{2048} - 1$?

(Proposed by Arsenii Nikolaiev)

Problem 3 Find the largest k for which there exists a permutation $(a_1, a_2, \ldots, a_{2022})$ of integers from 1 to 2022 such that for at least k distinct i with $1 \le i \le 2022$ the number $\frac{a_1+a_2+\ldots+a_i}{1+2+\ldots+i}$ is an integer larger than 1.

(Proposed by Oleksii Masalitin)

Problem 4 Let $ABCD$ be the cyclic quadrilateral. Suppose that there exists some line l parallel to BD which is tangent to the inscribed circles of triangles ABC, CDA. Show that l passes through the incenter of BCD or through the incenter of DAB.

(Proposed by Fedir Yudin)