Lexington Math Tournament - Theme Round

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by parmenides51, mathtastic

2010.1 Cheetahs

– Cheetahs in my dresser,
Cheetahs in my hair.
Cheetahs in my pants,
Cheetahs everywhere!
-A poem by J. Samuel Trabucco, Esq.

p1. J has several cheetahs in his dresser, which has 7 drawers, such that each drawer has the same number of cheetahs. He notices that he can take out one drawer, and redistribute all of the cheetahs (including those in the removed drawer) in the remaining 6 drawers such that each drawer still has an equal number of cheetahs as the other drawers. If he has at least one cheetah, what is the smallest number of cheetahs that he can have?

p2. J has 53 cheetahs in his hair, which he will put in 10 cages. Let A be the number of cheetahs in the cage with the largest number of cheetahs (there could be a tie, but in this case take the number of cheetahs in one of the cages involved in the tie). Find the least possible value of A.

p3. J has **98** cheetahs in his pants, some of which are male and the rest of which are female. He realizes that three times the number of male cheetahs in his pants is equal to nine more than twice the number of female cheetahs. How many male cheetahs are in his pants?

p4. Because J's cheetahs are everywhere, they are now running away. A particularly unintelligent one starts to run in a 720 mile loop at 80 miles per hour. J immediately starts to chase after it, starting at the same point, at 10 miles per hour at $12:00$ PM, but realizes one hour later that it would be more wise to turn around and run in the opposite direction in the loop, so he does this. Assuming both maintain a constant speed, at what time do J and the cheetah collide? Round to the nearest minute, and be sure to include AM or PM.

p5. Once J and his cheetah collide, J dies a very slow and painful death. The cheetahs come back for his funeral, which is held in a circular stadium with 10 rows. The first row has 10 seats in a circle, and each subsequent row has 3 more seats. However, no two adjacent seats may be occupied due to the size of the cheetahs. What is the maximum number of cheetahs that can fit in the stadium?

PS. You should use hide for answers.

2010.2 Lasers

– Laser beams are known for reflecting off solid objects. Whenever a laser beam hits a straight, solid wall, it reflects off in the opposite direction, at an angle to the wall that is equal to the angle at which it hits, as shown.
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png

The path of the laser L meets the wall, W, at an angle of 55.15 degrees coming from the right. Then, L reflects off W to a redirected path, L' , at an angle of 55.15 degrees going off to the left. In particular, if L were to meet W at an angle of 90 degrees, then L' would follow L, going backwards (straight up).

p6. Given a square $ABCD$, with $AB = 1$, mark the midpoints M and N of AB and BC, respectively. A laser beam shot from M to N, and the beam reflects of BC, CD, DA, and comes back to M. This path encloses a smaller area inside square $ABCD$. Find this area.

p7. Given a rectangle $EFGH$, with $EF = 3$ and $FG = 40$, mark a point P on FG such that $FP = 4$. A laser beam is shot from E to P, which then reflects off FG, then EH, then FG, etc. Once it reaches some point on GH, the beam is absorbed; it stops reflecting. How far does the beam travel?

p8. Given a rectangle $EFGH$ with $EF = 3$ and $FG = 2010$, mark a point P on FG such that $FP = 4$. A laser beam is shot from E to P, which then reflects off FG, then EH, then FG, etc. Once it reaches some point on GH, the beam is absorbed; it stops reflecting. How far does the beam travel?

p9. Given a triangle XYZ with $\angle Y = 90^o$, $XY = 1$, and $XZ = 2$, mark a point Q on YZ such that $\frac{ZQ}{ZY} = \frac{1}{3}$. A laser beam is shot from Q perpendicular to YZ, and it reflects off the sides of XYZ indefinitely. How many bounces does it take for the laser beam to get back to Q for the first time (not including the release from Q and the return to Q)?

p10. Given a triangle XYZ with $\angle Y = 90^o$, $XY = 1$, and $XZ = 2$, mark a point Q on YZ such that $\frac{ZQ}{ZY} = \frac{1}{3}$. A laser beam is shot from Q perpendicular to YZ, and it reflects off the sides of XYZ indefinitely. How far has the laser traveled when it reaches its 2010th bounce?

PS. You should use hide for answers.

2010.3 Rock Paper Scissors

– In the game of Rock Paper Scissors, typically two opponents make, with their hand, either a rock, a piece of paper, or a pair of scissors. If the two players play rock and scissors, the one who plays rock wins. If they play scissors and paper, the one who plays scissors wins, and if they play paper and rock, the one who plays paper wins. If they play the same thing, the result is a tie.

p11. Al and Bob are playing Rock Paper Scissors. Al plays rock. What is the probability that Al wins, given that Bob plays randomly and has an equal probability of playing rock, paper, and scissors?

p12. In a given game, what is the probability of a tie, given that both players play randomly and with an equal probability of playing rock, paper, and scissors?

p13. Al and Bob play Rock Paper Scissors until someone wins a game. What is the probability that this happens on the sixth game?

p14. Al and Bob are joined by Carl and D'Angelo, and they decide to play a team game of Rock Paper Scissors. A game is called perfect if some two of the four play the same thing, and the other two also play the same thing, but something different. For example, an example of a perfect game would be Al and Bob playing rock, and Carl and D'Angelo playing scissors, but if all four play paper, we do not have a perfect game. What is the probability of a perfect game?

p15. Al is bored of Rock Paper Scissors, and wants to invent a new game: $Z-Y-X-W-V$. Two players, each choose to play either Z, Y, X, W, or V. If they play the same thing, the result is a tie. However, Al must come up with a 'pecking order', that is, he must decide which plays beat which. For each of the 10 pairs of distinct plays that the two players can make, Al randomly decides a winner. For example, he could decide that W beats Y and that Z beats X, etc. What is the probability that after Al makes all of these 10 choices, the game is balanced, that is, playing each letter results in an equal probability of winning?

PS. You should use hide for answers.

2011.1 Interdisciplinary

– **p1.** Knowing very little about history, Bill assumes that the Hundred Years' War lasted exactly 100 years. Given that the war actually lasted for 116 years (from 1337 to 1453), what is the absolute value of the percent error of Bill's assumption? Express your answer as a decimal to the nearest tenth.

p2. The Epic Cycle of Ancient Greek poems includes five poems, the Cypria, the Aethiopis, the Iliupersis, the Nosti, and the Telegony. Each of these poems consists of some number of books of verse. The average number of books making up the last four poems is 3.5. When the Cypria is included among them, the average number of books among the five poems is 5. How many books of verse make up the Cypria?

p3. Zaroug is taking a test and must select the four noble gases from the set Helium, Argon, Boricon, Neon, Radium, Xenon and then arrange them in order of increasing mass. However, he only knows that there are exactly four noble gases in the set and that all six elements have different masses. Thus, he picks four elements from the set at random and then arranges them randomly. What is the probability he picks the four noble gases and orders them correctly?

p4. Let $ABCD$ be a rectangle with $AB = 4$ and $BC = 12$. A ball is fired from vertex A towards side BC at 10 units per second and bounces off a point E on BC such that $BE/BC = 1/4$. Every time the ball bounces off a wall, it loses energy in such a way that its speed is halved instantly. In addition, the angle at which the ball comes in is equal to the angle at which the ball bounces off the wall. Given that the ball only loses speed at the bounce points, how many seconds does it take the ball to reach vertex D?

p5. When two resistors in a circuit are connected in parallel, one of resistance m and one of resistance n, the equivalent resistance of the circuit R is given by $\frac{1}{R} = \frac{1}{m} + \frac{1}{n}$. How many ordered pairs (m, n) of positive integers are such that the equivalent resistance of a circuit with two parallel resistors of resistances m and n is equal to 10?

PS. You should use hide for answers.

2011.2 Schafkopf

– Schafkopf is a five-player game that has its roots in Germany - it is now popular (sometimes under the name Sheepshead) in locales such as Wisconsin, Canada/USA Mathcamp, and economics classes at Lexington High School. It is played with a standard deck of playing cards, with all cards between 2 and 6, inclusive, removed - there are thus four each of $7, 8, 9, 10$, Jack, Queen, King, and Ace. However, the suits differ from the standard - all queens, jacks, and diamonds form the "trump" suit, and the $7, 8, 9, 10$, King, and Ace in clubs, hearts, and spades form the three other suits.

p6. A card is drawn at random from a standard deck of 52 cards. What is the probability that it is used in the game of Schafkopf?

p7. A card is drawn at random from a Schafkopf deck. What is the probability that it is part of the "trump" suit?

p8. In Schafkopf, 6 cards are dealt to each player, with the two left over forming the "blind". What is the probability that both cards in the blind are queens?

p9. Carl, J., Leon, Peter, and Ryan sit down at a circular table to play Schafkopf. If Peter does not want to sit next to J., in how many ways can the five of them sit down? Assume that each seat is numbered 1 through 5, and any two configurations in which someone is sitting in a different seat in the two configurations are distinct.

p10. Let the points S, C, H, A, F, K, O, P be the vertices of a cube of side length 2 such that S is directly above F, C is directly above K, H is directly above O, and A is directly above P. Square $SCHA$ is rotated 45 degrees about its center and in its plane to square $S'C'H'A'$. The vertices of square $S'C'H'A'$ are then each connected to the two closest vertices of square $FKOP$, creating a new polyhedron $S'C'H'A'FKOP$ whose side and top views are shown on the left and right, respectively. Find the volume of this polyhedron.
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PS. You should use hide for answers.

2011.3 Sophomore Eating Habits

– *Note: Strictly speaking, none of the questions actually feature sophomore eating habits.*

p11. Darwin is buying pies, which are flat and circular, to throw at people on April Fools'. He can either buy one large pie with radius 4 or a whole number of small pies with radius 2. How many small pies does Darwin need to buy such that the total area of the small pies is greater than one large pie?

p12. Hao is selling cookies at a 2 day math competition. On the first day, he sold some number of cookies and some number of ramen packs, and earned $\$14$. The next day, he sold twice as many cookies and three times as many ramen packs, and earned $\$37$. If cookies are worth $\$1$ each and ramen packs are worth $\$0.50$ each, how many ramen packs did Hao sell on the first day?

p13. Surya is going to serve curry turkey to his friends such that if he has a spherical turkey of radius x, he must cut it into the whole number of pieces numerically closest to $x/2011$. If Surya needs to serve 2011 guests with no extra pieces left over and so that each person gets exactly one piece, how many different possible integer values are there for the radius of the spherical turkey?

Note: At the competition, the wording stated that "everyone gets a piece"

p14. In Flatland, Peijin is packing a circular pizza of area 45π into a rhombus $ABCD$ such that the pizza is tangent to all four sides of the rhombus. It is given that $\angle A$ is obtuse and that the distance between the two tangency points closest to A is 6. Find the perimeter of the rhombus pizza box in simplest radical form.

p15. Malik is adding 2011 flower decorations to a circular cake for a wedding along 2011 spaces planned out around the circumference of the cake, numbered 1 through 2011 counterclockwise. He places the first flower in space 1 and then cycles n spaces counterclockwise to a new space. If the space is empty, he places a flower in it, and doesn't otherwise. He then continues to cycle n spaces counterclockwise regardless of whether he placed a flower or not. For example, if $n = 7$, then after placing the first flower, he moves to space $1 + 7 = 8$, places a flower, moves to space $8 + 7 = 15$, places a flower, and so on. For what value of n, between 1 and 2011 inclusive, will the third to last flower Malik places end up in space 6?

PS. You should use hide for answers.

2012.1 Dragons

– *In honor of the Year of the Dragon, we present to you dragons: good and bad dragons alike.*

p1. It is said that Dragonite can circle the globe in just sixteen hours. If this is true, and if the Earth is assumed to be a sphere with radius 4000 miles, then what is Dragonite's average speed, in miles per hour? Use 3.14 for π and assume that the altitude above the Earth's surface at which Dragonite flies is negligible.

p2. Toothless the fish-loving dragon is at $(0, 0)$ and wants to get back to his home, located at $(6, 6)$. There is a pond full of fish at $(2, 5)$. If Toothless decides to only fly east one unit or north one unit at a time, then how many paths can he take that will allow him to catch some fish before returning home?

p3. Smaug the Magnificent likes to steal treasure for his ever-increasing mountain of riches. Fortunately for him, a village near his home has 2012 pieces of treasure. On the first day, he steals one piece of treasure from the village. After that, on the kth day, where $k \geq 2$, he plunders k more pieces than the amount he stole the day before. At this rate, Smaug will have stolen all of the treasure on the nth day. Find n.

p4. Temeraire the studious dragon has three distinct math books, two distinct science books, and one history book that he wants to read in the next few months. If he decides not to read two books of the same subject consecutively, then in how many different orders can he read the six books?

p5. Saphira the dragon is on forbidden ground patrolled by her enemy Thorn. Sensing that Thorn is 50 miles east of her and flying west at 50 miles per hour, Saphira flies north at a fast and constant speed. If Thorn can sense Saphira's presence when the two dragons are less than 40 miles away from each other, then what is the minimum speed at which Saphira must fly in order to evade his detection?

PS. You should use hide for answers.

2012.2 Knights and Knaves

– In the game Knights and Knaves, each player is assigned the role of either a knight or a knave. A knight always tells the truth while a knave always lies.

p6. Five players are to be assigned roles such that there are more knights than there are knaves. In how many ways can this be done?

p7. Ali, Bob, Cam, Dan, and Eve play Knights and Knaves and have the following conversation. Who is (are) the knave(s)? (Note: Either includes both. For example, if both Bob and Dan are knaves, then Ali is still correct.)
Ali: Either Bob or Dan is a knave.
Bob: Either Ali or Eve is a knight.
Cam: Either Bob or Eve is a knave.
Dan: Either Ali or Cam is a knave.
Eve: Exactly two of us are knaves.

p8. Eight players sit evenly around a circular table. In how many ways can the players be assigned roles such that each player can say, "The player sitting directly across from me is a knight"?

p9. Sixteen players stand in a four-by-four grid and each players says, "Every player adjacent to me (but not necessarily those diagonally across from me) is a knave." What is the minimum possible number of knights in the group?

p10. Eighteen players stand in two rows of nine, one row strictly behind the other. In how many ways can the players be assigned roles such that each player can say, "At least two of the players adjacent to me (i.e. beside me in the same row or directly across from me in the other row) are knaves"?

PS. You should use hide for answers.

2012.3 Evaluate-athon

– Athletes have the marathon and the triathlon. Mathletes, on the other hand, have the integrateathon and the evaluate-athon. Since most middle-schoolers cannot integrate because they do not know calculus, we present to you the evaluate-athon. Good luck!

p11. Evaluate $3 + 7 + 11 + 15 + ... + 51$.

p12. Evaluate $123 \cdot 357 + 123 \cdot 644 + 432 \cdot 357 + 432 \cdot 644$.

p13. Evaluate $\left\lfloor \frac{2012^4}{2011^2} \right\rfloor$ (where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x).

p14. Evaluate
$$\binom{10}{10}\left(\frac{1}{2}\right)^0 + \binom{10}{8}\left(\frac{1}{2}\right)^2 + \binom{10}{6}\left(\frac{1}{2}\right)^4 + ... + \binom{10}{0}\left(\frac{1}{2}\right)^{10}$$
.

p15. Evaluate $\frac{1^{-3}+3^{-3}+5^{-3}+7^{-3}+...}{1^{-3}+2^{-3}+3^{-3}+4^{-3}+...}$

PS. You should use hide for answers.

2013.1 Apple Pi and Other Desserts

– "*In order to make an apple pie from scratch, you must first create the universe.*" – Carl Sagan

p1. Surya decides to sell gourmet cookies at LMT. If he sells them for 25 cents each, he sells 36 cookies. For every 4 cents he raises the price of each cookie, he sells 3 fewer cookies. What is the smallest possible price, in cents, that Surya's jealous friends can change each cookie to so that Surya has no customers?

p2. Three French sorbets and four Italian gelatos cost 144 dollars. Six Italian gelatos and nine Florida sundaes cost 117 dollars. Seven French sorbets and 1 Florida sundae cost 229 dollars. If Arul wants to buy one of each type of ice cream, how much will it cost him?

p3. We call a number *delicious* if it has exactly 4 factors and is divisible by 2. How many numbers less than 99 are delicious?

p4. Charlie has 6 apple pies that form a ring. He starts with the first pie and puts on 10 scoops of whipped cream. Then, he moves to the next pie and, with a $1/2$ chance for each case, puts on either one less or one more scoop of whipped cream than the amount he put on the previous pie. He continues this process until he puts some whipped cream on the sixth pie. What is the probability that the number of scoops of whipped cream on the sixth pie differs from that on the first pie by exactly 1?

p5. Hao has 32 ounces of pure lemonade in his juice box. On the first day, he drinks an ounce and then refills the juice box with an ounce of Hater-ade and mixes the mixture thoroughly. On the second day, he drinks 2 oz of the juice and then refills the juice box with 2 oz of Hater-ade. On the third day, he drinks 3 oz and then refills the juice box with 3 oz of Hater-ade, and so on. Hao stops when the juice box has no more lemonade in it. How many ounces of Hater-ade did Hao drink?

PS. You should use hide for answers.

2013.2 Game Theory

– "*You have to learn the rules of the game. And then you have to play better than anyone else.*" – Albert Einstein

p6. Three standard six-sided dice are rolled and the values on the top faces are added. Let p be the probability of getting a total sum of 17 and q be the probability of getting a total sum of 18. Find p/q.

p7. Bill, Bob, and Ben write their favorite numbers on a sheet of paper. Bob points out that each number has its digits sum to 4, Bill points out that they are all divisible by the same prime number greater than 10, and Ben points out that none of them have 0's. Find the sum of Bill's, Bob's, and Ben's favorite numbers, given no two of them have the same favorite number.

p8. Sara has a 3×3 square tiled with alternating black and white colors (like a checkerboard). When she chooses a square, that square and the squares that share an edge with it switch colors. What is the minimum number of squares Sara needs to choose in order to cover the board with one color?

p9. A checker piece of radius 1 is placed at an arbitrary location on an infinitely large checkerboard of 1×1 squares . The checker piece covers n squares, either partially or completely. How many possible values for n are there?

p10. In a game of laser tag, a robot is in the center of a rectangular room with dimensions 6 meters by 8 meters. The robot has two perpendicular arms that can fire lasers straight outwards. The robot can fire the two lasers so long as they would hit the same wall if nothing blocked them. If a person is standing at a random location inside the room, what is the probability that she can be hit by a laser? The lasers do not bounce off walls.

PS. You should use hide for answers.

2013.3 Dream Jobs

– "*Choose a job you love, and you will never have to work a day in your life.*" – Confucius

p11. Darwin has a fruit stand at the market. Frank purchases 5 apples, and after a 5% tax, the price is $\$2.50$. Next, Rohil buys 3 apples. Assuming that everyone pays the 5% tax, how much did Rohil have to pay?

p12. In architecture class, Professor Radian wants to build a bridge over his circular pond with radius 4. He randomly chooses two points A and B on the circumference of the pond to be the endpoints of his new bridge. What is the probability that the length of the bridge is greater than $4\sqrt{3}$?

p13. Track stars Noah and Jonah run around a circular track. It takes Noah 3 minutes 20 seconds to run around the track, while it takes Jonah 3 minutes 45 seconds to run around the track. They start running around the track from the same spot going in the same direction. In how many minutes will the two of them be in the same spot along the track?

p14. Dan the Detective needs to decrypt a secret message intercepted from a clan of first graders. There are only 6 letters in the clan's alphabet, and Dan knows that the encryption code used by the clan is created by replacing every letter with another. For example, one encryption code may be $A \to B \to C \to D \to E \to F \to A$, where $A \to B$ means that every instance of A is replaced by a B. Note that a letter cannot be replaced by itself and no letter can replace multiple other letters. Given these conditions, how many different encryption codes can exist?

p15. The High Guardians of LHS have been assigned to protect a new, high priority room. A High Guardian's field of vision has a 360^o range so that it can see everything not directly blocked by a wall – in other words, one cannot see around walls – but once placed in the room, a High Guardian cannot move from that position. The room has 8 straight walls along its boundary and needs to be guarded so that every point in the entire room is in the field of vision of at least one High Guardian. What is the minimum number of High Guardians necessary for this task, no matter how the room is shaped?

PS. You should use hide for answers.

2014.1 Minions

– *"You have been beaten by a Gru"*

p1. All minions either have 1 or 2 eyes, and have one of 4 possible hairstyles. They are all thin and short, fat and short, or thin and tall. Gru doesn't want to have any 2 minions that look exactly alike, so what is the maximum possible amoount of minions can he get?

p2. Dave is shaped like two hemispeheres placed on either side of a vertical cylinder. If Dave is 4 feet across in width (excluding arms) , and has total volume 20π feet^3 , then how tall is he? (excluding legs and hair)

p3. There are 400 minions. 120 like Agnes, 160 like Margo, and 180 like Edith. 37 of the minions like both Agnes and Margo, and 74 minions like only Margo and Edith. If the number of minions that like all three girls to the number of minions that like Agnes and Edith to the number of minions that like none of the girls is $1 : 2 : 3$, how many minions like all 3 girls?

p4. There is a circular chair, that has the same radius as a minion, so to seat him perfectly on the chair. Gru wants to save space, so he starts shooting them with the shrinking ray from the first movie. If the minions are $1/5$ the size that they used to be (in diameter), how many minions can fit per chair?

p5. Minion A fires a laser from a corner of a room to the opposite corner. The room has length 32 and width 24. Minion B walks from the middle of the width side on one end of the room to as far as he can to the other side of the room along the length of the room. He stops right before he gets hit by the laser. What is the area of the total amount of area that the minion can walk over if he only walks left and right along the length of the room? The minion's head from above angle has radius 5.

PS. You should use hide for answers.

2014.2 Voting and political systems

– [i]"All those in favor of real numbers say '$\sqrt{-1}$' "[/i]

p1. In the Math Club, 30% of the people vote for algebra, $2/17$ voted for geometry and$1/16$ voted for combinatorics, and the rest voted for trigonometry. What is the smallest amount of people who could have voted in this election.?

p2. Currently, there are 9 political parties in Numerica, and 100 members of the senate. The political parties are going to change the number of senate members. "Demicrats" want the number to be a multiple of 2, "Triplicrats" want it to be divisible by 3, and so on, up to "Decacrats", who want it to be divisible by 10. They will hold a vote, and the proposed number of senate members will pass if at least 5 of the 9 parties approve of the number. If the political parties don't want to hire any new senate members, (so they must have less than or equal to 100 members), how many possible numbers of senate members will pass the vote?

p3. There are three candidates at the vertices of triangle ABC, which has a person at every point inside of it. Each person votes for the closest person to them, (and if there is a tie, restrains from voting). If

$$AB : AC : BC = 5 : 12 : 13,$$

then what is the ratio of the amount of votes for A to B to C?

p4. 3 people are on a ballot, and there are 10 voters. If each voter sees the previous voter's ballot, and each voter does not want to vote for the same person as the voter before him, how many ways are there for there to be a winner (with no ties for first place), who has a total of 5 votes for him?

p5. There are 7 people who are voting amongst themselves for who should become president (These 7 people are themselves the candidates). None of them can vote for themselves, but can vote for any of the other 6 people. In how many ways can there be a seven way tie, assuming the people are distinguishable?

PS. You should use hide for answers.

2014.3 Making the LMT

– *"Should we make an LMT "making the LMT" theme for the LMT theme round, or a "making the "making the LMT" theme for the LMT theme round" theme for the LMT theme round?"*

p1. The LMT consists of an individual round, a theme round, a team round, and a guts round. There are 20 problems on the LMT individual round. There are 3 themes with 5 problems a piece on the theme round. There are 10 potpourri problems and 1 proof problem on the team round, and 12 rounds of 3 problems on the guts round. How many problems are there in total?

p2. In LMT, problem writers can be slackers or workers. Slackers write 3 questions per day, and workers write 7 questions per day. If in one day, 94 questions are written and the only writers of questions are either slackers or workers, how many possible numbers of people could have written questions that day?

p3. Note: It is strongly recommended that you read and solve Problem 2 on this theme before solving this problem. Alan, who wrote the previous problem for the LMT, accidentally mis-wrote exactly one digit because his handwriting is so bad. If the intended answer was supposed to be 6, what is the sum of the digit that he was supposed to write and the digit that he did write instead?

p4. In the annual Lazy Mathematicians Together (LMT) conference, Bob decided to use his calculator to calculate 248 by typing in "$2 \cdot 2 \cdot 2 \cdot \cdot \cdot$", for a total of 48 2's. However, Bob's calculator has the property that whenever it tries to perform the operation $a \cdot b$, it gives back $a \cdot b - 1$. Bob's calculator eventually gives an answer of N, which can be written as $N = 2^{a_1}3^{a_2}5^{a_3}7^{a_4}M$, where M is an integer not divisible by any of $2, 3, 5, 7$. Find $2^{a_1} + 3^{a_2} + 5^{a_3} + 7^{a_4}$

p5. The graders for this problem are really lazy, and will grade this problem as correct even if the digits of the answer (all of which are unique and nonzero) are put in the wrong order (really!). Assuming that there is more than one digit and there is at least one prime number which the graders will mark correctly, what is the average of all answers which the graders will mark as correct to this problem?

PS. You should use hide for answers.

2015.1 Songs

– **p1.** Meghan Trainor is all about those base-systems. If she weighs 451 pounds in some base, and 127 pounds in a base that is twice as big, how much does she weigh in base 10?

p2. Taylor Swift made the song 15 in 2008, and the song 22 in 2012. If she had continued at the same rate, in what year would her song 1989 have come out?

p3. Sir Mix-A-Lot likes big butts and cannot lie. He is with his three friends, who all always lie. If each of the following statements was said by a different person, who is Sir Mix-A-Lot?
A: B and D both like big butts.
B: C and D either both like big butts, or neither of them do.
C: B does not like big butts.
D: A or C (or both) like big butts.

p4. Mark Ronson is going to give some uptown funk-tions to you. Bruno Mars is at 1 degree Fahrenheit right now (that is, time, in seconds, $t = 0$), and his heat h(t) is given by $h(t) = 2 \cdot h(t-1) + t$. If he will explode when he is over $1,000,000$ degrees Fahrenheit, at what time t will Bruno Mars explode?

p5. Jay-Z, who is a emcee, took 4 AMC contests, for a total of 100 problems. He got 99 problems correct and 1 problem wrong. His answer, z, to the problem he missed, was the square root of the correct 4-digit answer, j. If z is also 3 times the sum of the digits of j, then what is the ordered tuple (j, z)?

PS. You had better use hide for answers.

2015.2 Physics

– **p1.** Two buildings are connected by a rope. The rope is 20 meters long and each end is connected to the top corner of a building. If the rope droops 10 meters below the roofs of the buildings, then how far away are the buildings from each other?

p2. A ball bounces up and down and bounces up 80% of its height after each bounce. If it is dropped from a height of 2 feet above the ground, what is the total distance the ball will travel before coming to rest on the ground?

p3. The formula for the gravitational acceleration on a planet is $\frac{Gm}{r^2}$ where G is a constant, m is the mass of the planet, and r is the radius of the planet. If a certain planet has half the radius of Earth, and a third of the mass, what is the ratio of Earth's gravitational constant to that planet's?

p4. Albert knows that there are 6 types of leptons and 6 types of quarks, but he doesn't know the name of any of them. On a physics test, Albert is given a list of the names of all 12 particles, and has to label 6 as leptons and 6 as quarks. What is the probability that he gets at least 10 of the 12 particles correctly on the test?

p5. 3 planets are orbiting around a sun in coplanar orbits. One planet makes 1 orbit every 21 years, one makes an orbit every 33 years, and one makes an orbit every 77 years. If all of the planets and the sun lie along the same line right now, how long, in years, will it be before all 4 lie along the same line again?

PS. You had better use hide for answers.

2015.3 Frisbees

– In the following problems, some number of kids are evenly spaced around the perimeter of a unit circle. One frisbee is being thrown from kid to kid, taking the path of the line in between them. To throw the frisbee "5 kids left", for example, would mean to throw it in a straight line from the current kid to one 5 spaces in the clockwise direction, skipping over 4 kids.

p1. There are 2015 kids, and each time the frisbee is thrown, it is thrown 100 kids left. How many kids will be guaranteed never be able to touch the frisbee?

p2. Call the probability that a pass made by a certain player is not caught, that player's "Likelihood of a Missed Throw", or LMT for short. If Ivan, who is standing in a circle with 9 other players, has an LMT of 90%, and each player's LMT is the average of the LMTs of the people on their left and right, then what is the likelihood that the frisbee will make it all the way back around to Ivan?

p3. If Zach threw the frisbee to his friend Clive, who is standing 3 spaces to Zach's left, the frisbee would go half as far as if he threw it to Henry, who is standing 9 spaces to Zach's left. How far would the frisbee go if Clive were to throw it to Henry?

p4. Throwing the frisbee a distance d takes a total time of d^2 seconds. Moreover, after a person catches a frisbee, it takes them one second before they can throw it. Suppose Steven is in a circle with 12 total people, and starts by throwing the frisbee. It goes clockwise around the circle back to Steven in N throws, with him catching it after 11 seconds. What is the sum of all the possible values of N?

p5. If there are 187 kids in the circle, and each kid will either pass the frisbee 17 spaces to their left, or 11 spaces to their right, then how many ways are there to pass the frisbee 187 times, such that every person has thrown the frisbee once and every person has caught the frisbee once?

PS. You had better use hide for answers.

2016.1 Move-y

1 Memories all must have at least one out of five different possible colors, two of which are red and green. Furthermore, they each can have at most two distinct colors. If all possible colorings are equally likely, what is the probability that a memory is at least partly green given that it has no red?

[i]Proposed by Matthew Weiss

2 Mater is confused and starts going around the track in the wrong direction. He can go around 7 times in an hour. Lightning and Chick start in the same place at Mater and at the same time, both going the correct direction. Lightning can go around 91 times per hour, while Chick can go around 84 times per hour. When Lightning passes Chick for the third time, how many times will he have passed Mater (if Lightning is passing Mater just as he passes Chick for the third time, count this as passing Mater)?

[i]Proposed by Matthew Weiss

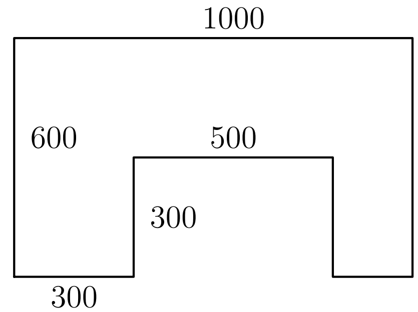
3 Geri plays chess against himself. White has a 5

[i]Proposed by Matthew Weiss

4 A male volcano is in the shape of a hollow cone with the point side up, but with everything above a height of 6 meters removed. The resulting shape has a bottom radius of 10 meters and a top radius of 7 meters, with a height of 6 meters. He sat above his bay, watching all the couples play. His lava grew and grew until he was half full of lava. Then, he erupted, lowering the height of the lava to 2 meters. What fraction of the lava remained in the volcano?

[i]Proposed by Matthew Weiss

5 Pixar Prison, for Pixar villains, is shaped like a 600 foot by 1000 foot rectangle with a 300 foot by 500 foot rectangle removed from it, as shown below. The warden separates the prison into three congruent polygonal sections for villains from The Incredibles, Finding Nemo, and Cars. What is the perimeter of each of these sections?



[i]Proposed by Peter Rowley

2016.2 Func-y

6 How many functions $f:\{1,2,3,4\}\to\{1,2,3\}$ are surjective?

[i]Proposed by Nathan Ramesh

7 Let $R(x)$ be a function that takes a natural number as input and returns a rectangle. $R(1)$ is known to have integer side lengths. Let $p(x)$ be the perimeter of $R(x)$ and let $a(x)$ be the area of $R(x)$. Suppose that $p(x+5) = 6p(x)$ for all x in the domain of R and that $a(x+2) = 12a(x)$ for all $x > 6$ in the domain of R. For $x \leq 6$, $a(x+1) = a(x)+2$. Suppose $p(16) = 1296$, and let the side lengths of $R(11)$ be a and b with $a \leq b$. Find the ordered pair (a, b).

[i]Proposed by Matthew Weiss

8 Consider the function $f : [0, 1) \to [0, 1)$ defined by $f(x) = 2x - \lfloor 2x \rfloor$, where $\lfloor 2x \rfloor$ is the greatest integer less than or equal to $2x$. Find the sum of all values of x such that $f^{17}(x) = x$.

[i]Proposed by Matthew Weiss

9 A function $f : \{1, 2, 3, \cdots, 2016\} \to \{1, 2, 3, \cdots, 2016\}$ is called *good* if the function $g(n) = |f(n) - n|$ is injective. Furthermore, a good function f is called *excellent* if there exists another good function f' such that $f(n) - f'(n)$ is nonzero for exactly one value of n. Let N be the number of good functions that are not excellent. Find the remainder when N is divided by 1000.

[i]Proposed by Nathan Ramesh

10 Let $S = \{1, 2, 3, 4, 5, 6\}$. Find the number of bijective functions $f : S \to S$ for which there exist exactly 6 bijective functions $g : S \to S$ such that $f(g(x)) = g(f(x))$ for all $x \in S$.

[i]Proposed by Nathan Ramesh

2016.3 Torn-y

11 A single elimination tournament is held with 2016 participants. In each round, players pair up to play games with each other. There are no ties, and if there are an odd number of players remaining before a round then one person will get a bye for the round. Find the minimum number of rounds needed to determine a winner.

[i]Proposed by Nathan Ramesh

12 A round robin tournament is held with 2016 participants. Each round, after seeing the results from the previous round, the tournament organizer chooses two players to play a game with each other that will result in a win for one of the players and a loss for the other. The tournament organizer wants each person to have a different total number of wins at the end of k rounds. Find the minimum possible value of k for which this can always be guaranteed.

[i]Proposed by Nathan Ramesh

13 A round robin tournament is held with 2016 participants. Each player plays each other player once and no games result in ties. We say a pair of players A and B is a *dominant pair* if all other players either defeat A and B or are defeated by both A and B. Find the maximum number dominant pairs.

[i]Proposed by Nathan Ramesh

14 A ladder style tournament is held with 2016 participants. The players begin seeded $1, 2, \cdots 2016$. Each round, the lowest remaining seeded player plays the second lowest remaining seeded player, and the loser of the game gets eliminated from the tournament. After 2015 rounds, one player remains who wins the tournament. If each player has probability of $\frac{1}{2}$ to win any game, then the probability that the winner of the tournament began with an even seed can be expressed has $\frac{p}{q}$ for coprime positive integers p and q. Find the remainder when p is divided by 1000.

[i]Proposed by Nathan Ramesh

15 A round robin tournament is held with 2016 participants. Each player plays each other player once and exactly one game results in a tie. Let W be the sum of the squares of each team's win total and let L be the sum of the squares of each team's loss total. Find the maximum possible value of $W - L$.

[i]Proposed by Matthew Weiss

2017.1 Building

– **p1**. Bob wants to build a bridge. This bridge has to be an arch bridge which reaches down on each side 10 feet and crosses a 20 foot gap. If the arch is shown by a semi circle with radius 9, and the bridge is 5 feet wide, how many cubic feet of material does Bob need?

p2. Evan constructs a ”poly-chain” by connecting regular polygons of side length 1 and having each adjacent polygon share a side. Additionally, Evan only creates a ”poly-chain” if the polygons in the chain all have a consecutive number of side lengths. For each ”poly-chain”, Evan then assigns it an ordered pair (m, n), where m is the number of polygons in the ”poly-chain”, and n is the number of sides of the largest polygon. Find all ordered pairs (m, n) that correspond to a ”poly-chain” with perimeter 17.

p3. Ben constructs a triangle ABC such that if M is the midpoint of BC then $AM = BC = 10$. Find the sum of all possible integer valued perimeters of $\triangle ABC$.

p4. Jason is creating a structure out of steel bars. First, he makes a cube. He then connects the midpoints of the faces to form a regular octahedron. He continues by connecting the midpoints of the faces of this octahedron to form another, smaller cube. Find the ratio of the volume of the smaller cube to the volume of the larger cube.

p5. Suppose Ben builds another triangle $\triangle ABC$ which has sidelengths $AB = 13$, $BC = 14$, $CA = 15$. Let D be the point of tangency between the incircle of $\triangle ABC$ and side BC, and let M be the midpoint of BC. The circumcircle of $\triangle ADM$ intersects the circumcircle of $\triangle ABC$ at a point $P \neq A$. If AP intersects BC at Q, find the length of BQ.

PS. You had better use hide for answers.

2017.2 Music

– **p6.** Generally, only frequencies between 20 Hz and $20,000$ Hz are considered audible. Nathan has a special LMT clarinet that only plays notes at integer multiples of 2017 Hz. How many different audible notes can Nathan play on his clarinet?

p7. Nathan's chamber group of 6 people have to line up to take a photo. They have heights of $\{62, 65, 65, 67, 69, 70\}$. They must line up left to right with the rule that the heights of 2 people standing next to each other can differ by at most 3. Find the number of ways in which this chamber group can line up from left to right.

p8. Mark, who loves both music and math, plays middle A on his clarinet at a frequency A_0. Then, one by one, each one of his m students plays a note one octave above the previous one. Using his math skills, Mark finds that, rounding to the nearest tenth, $\log_2 A_0 + \log_2 A_1 + ... + \log_2 A_m = 151.8$, where A_n denotes the frequency of the note n octaves above middle A. Given that $\log_2(A_0) = 8.8$, and that $A_n = A_0 \cdot 2^n$ for all positive integers n, how many students does Mark have?

p9. Janabel numbers the keys on her small piano from 1 to 10. She wants to choose a quintuplet of these keys (a, b, c), such that $a < b < c$ and each of these numbers are pairwise relatively prime. How many ways can she do this?

p10. Every day, John practices oboe in exact increments of either 0 minutes, 30 minutes, 1 hour, 1.5 hours, or 2 hours. How many possible ways can John practice oboe for a total of 5 hours in the span of 5 consecutive days?

PS. You had better use hide for answers.

2017.3 Games

– **p11.** A deck of cards contains 4 suites with 13 numbers in each suit. Evan and Albert are playing a game with a deck of cards. First, Albert draws a card. Evan wins if he draws a card with the same number as Albert's card or the same suit as Albert's card. What is the probability that Evan wins?

p12. 2 squares in a square grid are called *adjacent* if they share a side. In the game of minesweeper, we have a 2017×2017 grid of squares such that each square adjacent to a square which contains a mine is marked (A square with a mine in it is not marked). Also, every square with a mine in it is adjacent to at least one square without a mine in it. Given that there are $5,000$ mines, what is the difference between the greatest and least number of marked squares?

p13. There are 2016 stones in a pile. Alfred and Bobby are playing a game where on each turn they can take either a or b stones from the pile where a and b are distinct integers less than or equal to 6. They alternate turns, with Albert going first, and the last person who is able to take a stone wins For example, if $a = 3$,$b = 6$ and after Alfred's turn there are 2 stones left, then Alfred wins because Bobby is unable to make a move. Let A represent the number of ordered pairs (a, b) for which Alfred has a winning strategy and B represent the number of ordered pairs (a, b) for which Bobby has a winning strategy. Find $A - B$.

p14. For a positive integer n define $f(n)$ to be the number of unordered triples of positive integers (a, b, c) such that
(a) $a, b, c \le n$, and
(b) There exists a triangle ABC with side lengths a, b, c and points D, E, F on line segments AB,BC,CA respectively such that AD,BE,CF all have integer side lengths and $ADEF$ is a parallelogram.
Evan and Albert play a game where they calculate $f(2017)$ and $f(2016)$. Find $f(2017) - f(2016)$.

p15. Two players A and B take turns placing counters in squares of an $1 \times n$ board, with A going first. Each turn, players must place a counter in a square does that not share an edge with any square that already has a counter in it. The first player who is unable to make a move loses. Find all $n \le 20$ for which A has a winning strategy.

PS. You had better use hide for answers.

2018S.1 Olympics

– **p1.** The number of ways to rearrange PYEONGCHANG OLYMPIC can be expressed as $\frac{L!}{(M!)^T}$. Compute $L + M + T$.

p2. How many ways are there to color each ring in the olympic logo with the colors blue, yellow, black, green or red such that no two rings that intersect are the same color?
`https://cdn.artofproblemsolving.com/attachments/0/b/e32579790b5d633419a86d0c15af4ccd69849`
`png`

p3. At the Winter Olympics, Belarus and the Czech Republic won B and C medals respectively. Altogether, they won 10 medals. When the number of medals the US won is divided by $B+C$ it leaves a remainder of 3 and when it is divided by BC it leaves a remainder of 2. What is the least possible number of medals the U.S. could have won?

p4. Let ℓ_1 denote the string CURLING. Let ℓ_{k+1} be the sequence formed by inserting the substring 'CURLING' between each consecutive letter of lk . For example,

$$\ell_2 = \textbf{C}CURLING\textbf{U}CURLING\textbf{R}CURLING\textbf{L}CURLING\textbf{I}CURLING\textbf{N}CURLING\textbf{G}.$$

What is the 1537th letter in ℓ_{2018}?

p5. Five spectators of an Olympic wrestling match each stand at a random point around the circumference of the circular ring. Find the probability that they are all contained within a 90 degree arc of the circle.

PS. You should use hide for answers.

2018S.2 Trivia

– **p6.** Randy is playing a trivia game with 6 questions. Each question has 3 answer choices and if he answers all 6 questions correctly, he wins 5000 dollars. What is the expected amount of money Randy will win?

p7. It has recently been proven that a sudoku puzzle requires at least 17 numbers to be uniquely solvable. We section a 4×4 grid of boxes into four 2×2 squares. In each square we place the number $1, 2, 3$, or 4. An arrangement is called *sudoku-like* if there is exactly one of $1, 2, 3, 4$ in each row, column, and 2×2 box. How many sudoku-like arrangements are there? An example is given below.
`https://cdn.artofproblemsolving.com/attachments/6/a/f3c94ba7f064932c3789ac31880e2170b9a4e`
`png`

p8. A *superperfect* number is a number n such that if $\sigma(n)$ denotes the sum of its factors, then $\sigma(\sigma(n)) = 2n$. For example, $\sigma(16) = 1+2+4+8+16 = 31$ and $\sigma(31) = 32$. A and B are distinct numbers such that they each have exactly 10 factors and $|A-B| = 1$. Find the minimum possible value of $A+B$.

p9. A trivia game awards up to 2018 dollars split evenly among all of its winners such that each winner gets the maximum possible integer number of dollars. In a particular game if one more person had won each winner would have gotten two fewer dollars. How many possible number of winners are there for this game?

p10. The number 0 didn't exist until 628 AD, when it was introduced by the Indian mathematician Brahmagupta. The concept of 0 did exist much before then.

Let $\oplus$ be a binary operator such that for any 3 real numbers a, b, c, we have

$$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$

and

$$a \oplus b \oplus c = 4abc - 2ab - 2bc - 2ac + a + b + c.$$

Find all possible values of $20 \oplus 18$.

PS. You should use hide for answers.

2018S.3 Star Wars

– **p11.** R2-D2 is trying to break into a room. He realizes that the code is the same as $A + B$, where A and B are two positive integers greater than 1 such that

$$20_A + 12_B = 19_{B+A}$$

and

$$20_A - 12_B = 11_{B-A},$$

where the A subscript means the number is in base A. What is the password?

p12. Tatooine is located at the point $(0, 4)$, Hoth is located at the point $(3, 5)$, Alderaan was located at the point $(2, 1)$ and Endor is located at $(0, 0)$. At what point should the Death Star stop in order to minimize its combined distance to each location?

p13. How many ways can Chancellor Palpatine place indistinguishable galactic credits on a 4 by 8 grid such that each row has exactly 3 credits and each column has exactly 6 credits?

p14. Rey and Kylo Ren are each flying through space. Rey starts at $(-2, 2)$ and ends at $(2, 2)$ and Kylo Ren starts at $(-1, 1)$ and ends at $(1, 1)$. They can only travel 1 unit in the x or y direction at a time. Additionally, each of them must go to the x-axis before going to their respective destinations and take the shortest path possible to do so. Finally, their paths never intersect at any point. How many distinct configurations of paths can they take?

p15. A triangle ABC is formed with A is the current location of Darth Vader's spaceship, B is the location of the Rebel Base, and C is the location of the Death Star. Let D, E, and F be the locations of the spaceships of Luke Skywalker, Han Solo, and Princess Leia, respectively. It is true that D is the foot of the A-altitude, E is the foot of the A-angle bisector, and F is the foot of the A-median. Suppose the 4 segments on BC (some possibly of length 0), when measured in light-years, form an arithmetic sequence (in any order). What is the largest possible value of $\frac{BC}{AB}$?

PS. You should use hide for answers.

2018F.1 Chemistry

– Chemistry. BOOM BOOM. Things explode, and if you're not careful, you might explode, too.

p1. The half life of a radioactive isotope is the time it takes for the isotope to decay too half of its original concentration. Francium-223 decays with a half life of 22 minutes. Determine how many minutes it takes for a sample of francium-223 to decay to 25% of its original concentration.

p2. Ezra is walking across the periodic table. Determine the number of ways he can walk from gallium, atomic symbol Ga, to neon, atomic symbol Ne, if he must walk through sulfur, atomic symbol S, and can only walk right or up.

`https://cdn.artofproblemsolving.com/attachments/1/6/ff219ad16c7a082e01d40ce13a6577bdcd01e`
`png`

p3. In organic chemistry, molecules can be represented as polygons, where vertices are atoms and edges are bonds. A certain molecule is a finite plane of continuous hexagons, such as the one shown below. If there are 32 atoms and 41 bonds, how many hexagons are in the molecule?

`https://cdn.artofproblemsolving.com/attachments/a/e/fd3f1a2a1cc9ad19c0440a160d678a7730e16`
`png`

p4. Steel is an alloy of iron and carbon. Four iron atoms can be represented as mutually tangent spheres, and the carbon atom can be represented as a sphere externally tangent to all four iron atoms. If the radius of the iron atom is 12 angstroms, determine the radius of a carbon atom in angstroms.

p5. Electrons in an atom are described by a set of four quantum numbers, $\{n, l, m_l, m_s\}$, according to the following restrictions: all quantum numbers must be integers except m_s, which can be either $+\frac{1}{2}$ or $-\frac{1}{2}$, $0 \le l < n$ and $|m_l| \le l$. Additionally, no two electrons can have the same set of four quantum numbers. Determine the number of electrons that can exist such that $n \le 20$.

PS. You should use hide for answers.

2018F.2 Mafia

– Mafia is a game where there are two sides: The village and the Mafia. Every night, the Mafia kills a person who is sided with the village. Every day, the village tries to hunt down the Mafia through communication, and at the end of every day, they vote on who they think the mafia are.

p6. Patrick wants to play a game of mafia with his friends. If he has 10 friends that might show up to play, each with probability $1/2$, and they need at least 5 players and a narrator to play, what is the probability that Patrick can play?

p7. At least one of Kathy and Alex is always mafia. If there are 2 mafia in a game with 6 players, what is the probability that both Kathy and Alex are mafia?

p8. Eric will play as mafia regardless of whether he is randomly selected to be mafia or not, and Euhan will play as the town regardless of what role he is selected as. If there are 2 mafia and 6 town, what is the expected value of the number of people playing as mafia in a random game with Eric and Euhan?

p9. Ben is trying to cheat in mafia. As a mafia, he is trying to bribe his friend to help him win the game with his spare change. His friend will only help him if the change he has can be used to form at least 25 different values. What is the fewest number of coins he can have to achieve this added to the fewest possible total value of those coins? He can only use pennies, nickels, dimes, and quarters.

p10. Sammy, being the very poor mafia player he is, randomly shoots another player whenever he plays as the vigilante. What is the probability that the player he shoots is also not shot by the mafia nor saved by the doctor, if they both select randomly in a game with 8 people? There are 2 mafia, and they cannot select a mafia to be killed, and the doctor can save anyone.

PS. You should use hide for answers.

2018F.3 Music

– What music do you like the best? Perhaps theme music from anime? Or maybe you like gaming theme music from YouTubing gamers on arcade games!

p11. Every note in a musical phrase that is 2 measures long lasts for $1/8$ of the measure of $1/4$

of it. How many different rhythms can the phrase have if there are no rests?

p12. I play a random chord of three different notes. It is called dissonant if the distance between any two of those notes are 1, 2, 6, 10, or 11 semitones apart. Given that the longest distance between two notes is at most 12 semitones, what is the probability that the chord will be dissonant?

p13. Two distinct notes are called an interval and if they are $0, 5$, or 7 semitones apart modulo 12, they are a perfect interval. (Modulo 12 means the remainder when divided by 12.) If a piano has 88 notes, and consecutive notes are 1 semitone apart, how many perfect intervals can be played?

p14. An ensemble has a violin, a viola, and a cello. A chord has 3 notes and each instrument can play 1 or 2 of them at a time. How many ways can they play the chord if every note in the chord must be played? (Octaves don't matter.)

PS. You should use hide for answers.

2019S.1 Pick Up Lines

– **p1.** Anka wants to know if it hurt when her crush fell from the sky. She curls up into a ball and jumps off a 80 meter high building. If she bounces up to $3/4$ of the previous height each bounce, how many times can she bounce while still moving a total distance of less than 300 meters?

p2. Alex wants to rearrange the alphabet to put him and his crush next to each other. If he randomly rearranges it, what is the probability that "u" and "i" are next to each other?

p3. Jeffrey, being from Tennessee, sees 10s everywhere he looks. If he assigns to each of his 10000 lovers a unique integer ID number from 1 to 10000 how many of them will have the sequence "10" in their ID?

p4. Andrew is getting lost in Amy's eyes, or more specifically, her i's: $a_i, b_i, c_i, ..., z_i$.
Let $a_n = \frac{1}{2^n}$,$b_n = \frac{1}{3^n}$, ..., $z_n = \frac{1}{27^n}$.
Additionally, let $S = \{(i_1, i_2, ..., i_{26})|i_1 \ge 1, i_2, i_3, ..., i_{26} \ge 0, i_1, i_2, ..., i_{26} \in Z\}$.
Find the sum of $a_{i_1} b_{i_2} ... z_{i_{26}}$ over all $(i_1, i_2, ..., i_{26}) \in S$.

p5. Janabel is in love with regular pentagon $ANGEL$ with side length 4 and area x. Find x^2.

PS. You should use hide for answers.

2019S.2 Goldilocks Act 1

– **p6.** Goldilocks is walking around in the magical forked forest. She has a $1/2$ chance of choosing the correct path at each fork. If she chooses the wrong path more than two times, she gets lost. What is the probability she gets lost if she encounters five forks?

p7. Goldilocks is being followed by a squirrel. Whenever the squirrel is 20 ft behind Goldilocks, it runs up until it catches up with her, then stays in place until Goldilocks is 20 ft ahead again. The squirrel runs at a rate of 10 ft/s and Goldilocks walks at a rate of 5 ft/s. If Goldilocks and the squirrel start at the same place, how many seconds will pass before the squirrel catches up to Goldilocks again?

p8. Goldilocks walks up to the door, and observes that there are ten locks. She looks under the mattress and finds ten keys. To open the door, every key must be in the correct lock. Moreover, if she puts one key in its correct lock, it cannot be removed anymore. If she tries a random combination of the remaining keys every minute, what is the expected number of minutes until she opens the door?

p9. A river runs parallel to Goldilocks' trail. Two congruent circular fields are tangent to the trail, the river, and each other. A third smaller field is externally tangent to the two fields and the river. If the distance between the river and the trail is 200 feet, what is the radius of the smaller field?

p10. Goldilocks stumbles upon a large house with an even larger field of flowers. She reads a sign that says, "Three bears in the house, 3^{2019} flowers in the field." If the flowers are arranged in rows of 100, how many flowers will be left over?

PS. You should use hide for answers.

2019S.3 Goldilocks Act 2

– **p11.** As the bears walk back from their home, they realize they left their fish by the river. The river is a circle centered at their home with radius 6 miles. If they are currently 4 miles away from their home, what is the shortest possible distance, in miles, that they must travel to get to the river and then return home?

p12. Goldilocks walks into the bedroom and sees three beds, B_1, B_2, and B_3. Sleepy from her meal, she decides to take a nap in each bed. She slept three times as long in B_2 as B_1 and ten minutes less in B_1 as B_3. If the most time she slept in any single bed was 2019 minutes, how long did she sleep in total?

p13. The Bears are on their way back from a stroll. They will return home at a random time between 3 and 4 pm, while Goldilocks will wake up at a random time between 2 and 4 pm. The bears will catch Goldilocks if they arrive home no more than 15 minutes after she wakes up. Find the probability that Goldilocks will escape without being caught.

p14. While sleeping in Mama Bear's bed, Goldilocks dreams about different colored sheep. She sees two blue sheep, two red sheep, and one green sheep. How many ways are there to arrange the sheep in a line if no two sheep of the same color can be standing next to each other? Note that sheep of the same color are indistinguishable.

p15. A bored and hungry Goldilocks finds an infinite number of raisins in her pocket. On the kitchen table lies an infinite number of bowls of porridge. She labels the bowls, numbering the leftmost bowl 1, the second leftmost bowl 2, and so on. She then distributes her raisins, starting by adding 1raisin to Bowl 1. Next, if she adds r raisins to Bowl b, then she will add r raisins to Bowl $2b$, and $r+1$ raisins to Bowl $2b+1$. Into how many of the 22019 leftmost bowls of porridge will Goldilocks add exactly 2017 raisins?

PS. You should use hide for answers.

2019F.1 Joe Quigley

– **This section was written in memory of Joseph "Joe" John IV Quigley, who passed away last October. Joe Quigley ran the Math Club in Lexington for over twenty-four years, making math accessible and fun for students of all ages and abilities. His love of math was only eclipsed by his love for teaching, and he will be greatly missed by the entire Lexington community for his humor, patience, and dedication to his students.**

p1. Joe Quigley writes the following expression on the board for his students to evaluate:

$$2 \times 4 + 3 - 1$$

However, his students have not learned their order of operations so they randomly choose which operations to perform first, second, and third. How many different results can the students obtain?

p2. Joe Quigley flies airplanes around the Cartesian plane. There are two fuel stations, one at $(8,12)$ and another at $(25,5)$. He must station his home on the x-axis, and wants to be the same distance away from both station. Compute this distance.

p3. Joe Quigley has 12 students in his math class. He will distribute N worksheets among the students. Find the smallest positive integer N for which any such distribution of the N worksheets among the 12 students results in at least one student having at least 3 worksheets.

p4. Joe Quigley writes the number $4^{4^{4^4}}$ on the board. He correctly observes that this number has $2^a + b$ factors, where a and b are positive integers. Find the maximum possible value of a.

p5. Joe Quigley is teaching his students geometric series and asks them to compute the value of the following series:
$$\sum_{x\ge 1}\frac{x(x+1)}{2^x}$$
Compute this value.

PS. You should use hide for answers.

2019F.2 Astronomy

– **p6.** Alex and Anka go to see stars. Alex counts thirteen fewer stars than Anka, and both of the numbers of stars they counted are square numbers. It is known that exactly eight stars were counted by both of them. How many distinct stars did they count in total?

p7. Three planets with coplanar, circular, and concentric orbits are shown on the backside of this page. The radii of the three circles are $3, 4$, and 5. Initially, the three planets are collinear. Every hour, the outermost planet moves one-sixth of its full orbit, the middle planet moves one-fourth of its full orbit, and the innermost planet moves one-third of its full orbit (A full orbit occurs when a planet returns to its initial position). Moreover, all three planets orbit in the same direction. After three hours, what is the area of the triangle formed by the planets as its three vertices?
`https://cdn.artofproblemsolving.com/attachments/3/2/b1f7cc5ee1108d5ba68d0464df18782243630`
`png`

p8. Planets X and Y are following circular orbits around the same star. It takes planet X 120hours to complete a full trip around the star, and it takes planet Y 18 hours to complete a full trip around the star. If both planets begin at the same location in their orbit and travel clockwise, how many times will planet Y pass planet X in the time it takes planet X to complete a full trip around the star?

p9. In a certain stellar system, four asteroids form a rectangle. Your spaceship lies in the rectangle formed. The distances between three of these asteroids and your space ship are 1 light minute, 8 light minutes, and 4 light minutes. If the distance between your space ship and the last asteroid is an integer number of light minutes away then how far away from the last asteroid is your space ship?

p10. Two lost lovers, Laxe and Kaan, are both standing on the equators of planets with radius 13 miles. The center of the planets are 170 miles apart. At some time, both of them are as close to each other as possible. The planets rotate in opposite directions of each other at the same rate. What is the maximum possible distance between Laxe and Kaan such that they are still able to see each other?

PS. You should use hide for answers.

2019F.3 Holidays

– **p11.** Festivus occurs every year on December 23rd. In 2019, Festivus will occur on a Monday. On what day will Festivus occur in the year 2029?

p12. Leakey, Marpeh, Arik, and Yehau host a Secret Santa, where each one of them is assigned to give a present to somebody other than themselves. How many ways can the gifting be assigned such that everyone receives exactly one gift?

p13. How many permutations of the word C HR I ST M AS are there such that the S's are not next to each other and there is not a vowel anywhere between the two S's?

p14. Dasher, Dancer, Prancer, Vixen, Comet, Cupid, Donner, Blitzen, and Rudolph (9 reindeer) are guiding Santa's sleigh. They are arranged in a 3×3 array. You, the elf, have a big responsibility. You must place Santa's reindeer in a manner so that all of Santa's requests are met: • Donner is forgetful and must be put in the back row so Santa can keep an eye on Donner. • Additionally, Rudolph's big red nose distracts Donner, so Rudolph and Donner cannot be in the same column. • Finally, Comet is the fastest and must be put in the front row.
How many options do you have for arranging Santa's reindeer?

p15. Marpeh has a Christmas tree in the perfect shape of a right circular cone. The tree has base radius 8 inches and slant height 32 inches. He wants to place 3 ornaments on the surface of the tree with the following rules: • The red ornament is placed at the top of the tree. • The yellow ornament is placed along the circumference of the base of the tree. • The blue ornament is placed such that it is the same distance from the red and yellow ornaments when traveling on the surface of the tree.
What is the furthest possible surface distance that the blue ornament could be from the red ornament?

PS. You should use hide for answers.

2022 F.1 Tetris

– Tetris is a Soviet block game developed in 1984, probably to torture misbehaving middle school children. Nowadays, Tetris is a game that people play for fun, and we even have a mini-event featuring it, but it shall be used on this test for its original purpose. The 7 Tetris pieces, which will be used in various problems in this theme, are as follows:
https://cdn.artofproblemsolving.com/attachments/b/c/f4a5a2b90fcf87968b8f2a1a848ad32ef5201
png

p1. Each piece has area 4. Find the sum of the perimeters of each of the 7 Tetris pieces.

p2. In a game of Tetris, Qinghan places 4 pieces every second during the first 2 minutes, and 2 pieces every second for the remainder of the game. By the end of the game, her average speed is 3.6 pieces per second. Find the duration of the game in seconds.

p3. Jeff takes all 7 different Tetris pieces and puts them next to each other to make a shape. Each piece has an area of 4. Find the least possible perimeter of such a shape.

p4. Qepsi is playing Tetris, but little does she know: the latest update has added realistic physics! She places two blocks, which form the shape below. Tetrominoes $ABCD$ and $EFGHIJ$ are both formed from 4 squares of side length 1. Given that $CE = CF$, the distance from point I to the line AD can be expressed as $\frac{A\sqrt{B}-C}{D}$. Find $1000000A + 10000B + 100C + D$.
https://cdn.artofproblemsolving.com/attachments/9/a/5e96a855b9ebbfd3ea6ebee2b19d7c0a82c7c
png

p5. Using the following tetrominoes:
https://cdn.artofproblemsolving.com/attachments/3/3/464773d41265819c4f452116c1508baa66078
png
Find the number of ways to tile the shape below, with rotation allowed, but reflection disallowed:
https://cdn.artofproblemsolving.com/attachments/d/6/943a9161ff80ba23bb8ddb5acaf699df187e0
png

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2022 F.2 World Cup

– Ephram Chun is a senior and math captain at Lexington High School. He is well-loved by the freshmen, who seem to only listen to him. Other than being the father figure that the freshmen never had, Ephramis also part of the Science Bowl and Science Olympiad teams along with being part of the highest orchestra LHS has to offer. His many hobbies include playing soccer, volleyball, and the many forms of chess. We hope that he likes the questions that we've dedicated to him!

p1. Ephram is scared of freshmen boys. How many ways can Ephram and 4 distinguishable freshmen boys sit together in a row of 5 chairs if Ephram does not want to sit between 2 freshmen boys?

p2. Ephram, who is a chess enthusiast, is trading chess pieces on the black market. Pawns are worth $\$100$, knights are worth $\$515$, and bishops are worth $\$396$. Thirty-four minutes ago, Ephrammade a fair trade: 5 knights, 3 bishops, and 9 rooks for 8 pawns, 2 rooks, and 11 bishops. Find the value of a rook, in dollars.

p3. Ephramis kicking a volleyball. The height of Ephram's kick, in feet, is determined by

$$h(t) = -\frac{p}{12}t^2 + \frac{p}{3}t,$$

where p is his kicking power and t is the time in seconds. In order to reach the height of 8 feet between 1 and 2 seconds, Ephram's kicking power must be between reals a and b. Find is $100a+b$.

p4. Disclaimer: No freshmen were harmed in the writing of this problem.

Ephram has superhuman hearing: He can hear sounds up to 8 miles away. Ephramstands in the middle of a 8 mile by 24 mile rectangular grass field. A freshman falls from the sky above a point chosen uniformly and randomly on the grass field. The probability Ephram hears the freshman bounce off the ground is $P\%$. Find P rounded to the nearest integer.
https://cdn.artofproblemsolving.com/attachments/4/4/29f7a5a709523cd563f48176483536a2ae656
png

p5. Ephram and Brandon are playing a version of chess, sitting on opposite sides of a 6×6 board. Ephram has 6 white pawns on the row closest to himself, and Brandon has 6 black pawns on the row closest to himself. During each player's turn, their only legal move is to move one pawn one square forward towards the opposing player. Pawns cannot move onto a space occupied by another pawn. Players alternate turns, and Ephram goes first (of course). Players take turns until there are no more legal moves for the active player, at which point the game ends. Find the number of possible positions the game can end in.

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2022 F.3 Ephram

– The World Cup, featuring 17 teams from Europe and South America, as well as 15 other teams that honestly don't have a chance, is a soccer tournament that is held once every four years. As we speak, Croatia andMorocco are locked in a battle that has no significance whatsoever on the winner, but if you would like live score updates nonetheless, feel free to ask your proctor, who has no obligation whatsoever to provide them.

p1. During the group stage of theWorld Cup, groups of 4 teams are formed. Every pair of teams in a group play each other once. Each team earns 3 points for each win and 1 point for each tie. Find the greatest possible sum of the points of each team in a group.

p2. In the semi-finals of theWorld Cup, the ref is bad and lets $11^2 = 121$ players per team go on the field at once. For a given team, one player is a goalie, and every other player is either a defender, midfielder, or forward. There is at least one player in each position. The product of the number of defenders, midfielders, and forwards is a mulitple of 121. Find the number of ordered triples (number of defenders, number of midfielders, number of forwards) that satisfy these conditions.

p3. Messi is playing in a game during the Round of 16. On rectangular soccer field $ABCD$ with $AB = 11$, $BC = 8$, points E and F are on segment BC such that $BE = 3$, $EF = 2$, and $FC = 3$. If the distance betweenMessi and segment EF is less than 6, he can score a goal. The area of the region on the field whereMessi can score a goal is $a\pi + \sqrt{b} + c$, where a, b, and c are integers. Find $10000a + 100b + c$.

p4. The workers are building theWorld Cup stadium for the 2022 World Cup in Qatar. It would take 1 worker working alone 4212 days to build the stadium. Before construction started, there were 256 workers. However, each day after construction, 7 workers disappear. Find the number of days it will take to finish building the stadium.

p5. In the penalty kick shootout, 2 teams each get 5 attempts to score. The teams alternate shots and the team that scores a greater number of times wins. At any point, if it's impossible for one team to win, even before both teams have taken all 5 shots, the shootout ends and nomore shots are taken. If each team does take all 5 shots and afterwards the score is tied, the shootout enters sudden death, where teams alternate taking shots until one team has a higher score while both teams have taken the same number of shots. If each shot has a $\frac{1}{2}$ chance of scoring, the expected number of times that any team scores can be written as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A + B$.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2023F 1A Sam dumps tea for 6 hours at a constant rate of 60 tea crates per hour. Eddie takes 4 hours to dump the same amount of tea at a different constant rate. How many tea crates does Eddie dump per hour?

Proposed by Samuel Tsui

2023F 2A On day 1 of the new year, John Adams and Samuel Adams each drink one gallon of tea. For each positive integer n, on the nth day of the year, John drinks n gallons of tea and Samuel drinks n^2 gallons of tea. After how many days does the combined tea intake of John and Samuel that year first exceed 900 gallons?

Proposed by Aidan Duncan

2023F 3A A rectangular tea bag $PART$ has a logo in its interior at the point Y . The distances from Y to PT and PA are 12 and 9 respectively, and triangles $\triangle PYT$ and $\triangle AYR$ have areas 84 and 42 respectively. Find the perimeter of pentagon $PARTY$.

Proposed by Muztaba Syed

2023F 4A Let *Revolution*$(x) = x^3+Ux^2+Sx+A$, where U, S, and A are all integers and $U+S+A+1 = 1773$. Given that *Revolution* has exactly two distinct nonzero integer roots G and B, find the minimum value of $|GB|$.

Proposed by Jacob Xu

2023F 5A Paul Revere is currently at (x_0, y_0) in the Cartesian plane, which is inside a triangle-shaped ship with vertices at $\left(-\frac{7}{25}, \frac{24}{25}\right)$, $\left(-\frac{4}{5}, \frac{3}{5}\right)$, and $\left(\frac{4}{5}, -\frac{3}{5}\right)$. Revere has a tea crate in his hands, and there is a second tea crate at $(0, 0)$. He must walk to a point on the boundary of the ship to dump the tea, then walk back to pick up the tea crate at the origin. He notices he can take 3 distinct paths to walk the shortest possible distance. Find the ordered pair (x_0, y_0).

Proposed by Derek Zhao

2023F 1B Evaluate $\binom{6}{0}+\binom{6}{1}+\binom{6}{4}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}$

Proposed by Jonathan Liu

2023F 2B A four-digit number n is said to be *literally 1434* if, when every digit is replaced by its remainder when divided by 5, the result is 1434. For example, 1984 is *literally 1434* because 1 mod 5 is 1, 9 mod 5 is 4, 8 mod 5 is 3, and 4 mod 5 is 4. Find the sum of all four-digit positive integers that are *literally 1434*.

Proposed by Evin Liang

2023F 3B Evin and Jerry are playing a game with a pile of marbles. On each players' turn, they can remove 2, 3, 7, or 8 marbles. If they can't make a move, because there's 0 or 1 marble left, they lose the game. Given that Evin goes first and both players play optimally, for how many values of n from 1 to 1434 does Evin lose the game?

Proposed by Evin Liang

2023F 4B]In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Let M be the midpoint of side AB, G be the centroid of $\triangle ABC$, and E be the foot of the altitude from A to BC. Compute the area of quadrilateral $GAME$.

Proposed by Evin Liang

2023F 5B Bamal, Halvan, and Zuca are playing *The Game*. To start, they're placed at random distinct vertices on regular hexagon $ABCDEF$. Two or more players collide when they're on the same vertex. When this happens, all the colliding players lose and the game ends. Every second, Bamal and Halvan teleport to a random vertex adjacent to their current position (each with probability $\frac{1}{2}$), and Zuca teleports to a random vertex adjacent to his current position, or to the vertex directly opposite him (each with probability $\frac{1}{3}$). What is the probability that when *The Game* ends Zuca hasn't lost?

Proposed by Edwin Zhao

2023F 1C How many distinct triangles are there with prime side lengths and perimeter 100?

Proposed by Muztaba Syed

2023F 2C Let R be the rectangle on the cartesian plane with vertices $(0,0)$, $(5,0)$, $(5,7)$, and $(0,7)$. Find the number of squares with sides parallel to the axes and vertices that are lattice points that lie within the region bounded by R.

Proposed by Boyan Litchev

2023F 3C Determine the least integer n such that for any set of n lines in the 2D plane, there exists either a subset of 1001 lines that are all parallel, or a subset of 1001 lines that are pairwise nonparallel.

Proposed by Samuel Wang

2023F 4C The equation of line ℓ_1 is $24x - 7y = 319$ and the equation of line ℓ_2 is $12x - 5y = 125$. Let a be the number of positive integer values n less than 2023 such that for both ℓ_1 and ℓ_2 there exists a lattice point on that line that is a distance of n from the point $(20, 23)$. Determine a.

Proposed by Christopher Cheng

2023F 5C In equilateral triangle ABC, $AB = 2$ and M is the midpoint of AB. A laser is shot from M in a certain direction, and whenever it collides with a side of ABC it will reflect off the side such that the acute angle formed by the incident ray and the side is equal to the acute angle formed by the reflected ray and the side. Once the laser coincides with a vertex, it stops. Find the sum of the smallest three possible integer distances that the laser could have traveled.

Proposed by Jerry Xu