Stanford Mathematics Tournament 2020

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by parmenides51

– Team Round

– **p1.** Find the sum of the largest and smallest value of the following function: $f(x) = |x - 23| + |32 - x| - |12 + x|$ where the function has domain $[-37, 170]$.

p2. Given a convex equiangular hexagon with consecutive side lengths of $9, a, 10, 5, 5, b$, where a and b are whole numbers, find the area of the hexagon.

p3. Let $\S = 1, 2, ..., 100$. Compute the minimum possible integer n such that, for any subset $T \subset S$ with size n, every integer a in S satisfies the relation $a \equiv bc \mod 101$, for some choice of integers b, c in T.

p4. Let C be the circle of radius 2 centered at $(4, 4)$ and let L be the line $x = -2$. The set of points equidistant from C and from L can be written as $ax^2 + by^2 + cxy + dx + ey + f = 0$ where a, b, c, d, e, f are integers and have no factors in common. What is $|a + b + c + d + e + f|$?

p5. If a is picked randomly in the range $\left(\frac{1}{4}, \frac{3}{4}\right)$ and b is chosen such that $int_a^b \frac{1}{x^2} dx = 1$, compute the expected value of $b - a$.

p6. Let $A_1, A_2, .., A_{2020}$ be a regular 2020-gon with a circumcircle C of diameter 1. Now let P be the midpoint of the small-arc $A_1 - A_2$ on the circumcircle C. Then find $\sum_{2020}^{i=1} |PA_i|^2$.

p7. A certain party of 2020 people has the property that, for any 4 people in the party, there is at least one person of those 4 that is friends with the other three (assume friendship is mutual). Call a person in the party a politician if they are friends with the other 2019 people in the party. If n is the number of politicians in the party, compute the sum of the possible values of n.

p8. Let S be an n-dimensional hypercube of sidelength 1. At each vertex draw a hypersphere of radius $1/2$, let Ω be the set of these hyperspheres. Consider a hypersphere Γ centered at the center of the cube that is externally tangent to all the hyperspheres in Ω. For what value of n does the volume of Γ equal to the sum of the volumes of the hyperspheres in Ω.

p9. Solve for C:

$$\frac{2\pi}{3}=\int_0^1 \frac{1}{\sqrt{Cx-x^2}}dx$$

p10. Nathan and Konwoo are both standing in a plane. They each start at $(0, 0)$. They play many games of rock-paper-scissors. After each game, the winner will move one unit up, down, left, or right, chosen randomly. The outcomes of each game are independent, however, Konwoo is twice as likely to win a game as Nathan. After 6 games, what is the probability that Konwoo is located at the same point as Nathan? (For example, they could have each won 3 games and both be at $(1, 2)$.)

p11. Suppose that x, y, z are real positive numbers such that $(1 + x^4y^4)e^z + (1 + 81e^{4z})x^4e^{-3z} = 12x^3y$. Find all possible values of $x + y + z$.

p12. Given a large circle with center (x_0, y_0), one can place three smaller congruent circles with centers (x_1, y_1),(x_2, y_2),(x_3, y_3) that are pairwise externally tangent to each other and all internally tangent to the outer circle. If this placement makes $x_0 = x_1$ and $y_1 > y_0$, we call this an "up-split". Otherwise, if the placement makes $x_0 = x_1$ and $y_1 < y_0$, we call it a "down-split." Alice starts at the center of a circle C with radius 1. Alice first walks to the center of the upper small circle C_u of an up-slitting of C. Then, Alice turns right to walk to the center of the upper-right small circle of a down-splitting of C_d. Alice continues this process of turning right and walking to the center of a new circle created by alternatingly up- and down-splitting. Alice's path will form a spiral converging to (x_A, y_A). On the otherhand, Bob always up-splits the circle he is in the center of, turns right and finds the center of the next small circle. His path will converges to (x_B, y_B). Compute $\left|\frac{1}{x_A}-\frac{1}{x_B}\right|$.

p13. Compute the sum of all natural numbers b less than 100 such that b is divisible by the number of factors of the base-10 representation of 2020_b.

p14. Iris is playing with her random number generator. The number generator outputs real numbers from 0 to 1. After each output, Iris computes the sum of her outputs, if that sum is larger than 2, she stops. What is the expected number of outputs Iris will receive before she stops?

p15. Evaluate

$$\int_0^{\frac{\pi}{2}} \ln(9\sin^2\theta + 121\cos^2\theta)d\theta$$

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Geometry Round

1 A circle with radius 1 is circumscribed by a rhombus. What is the minimum possible area of this rhombus?

2 Let $\triangle ABC$ be a right triangle with $\angle ABC = 90^o$. Let the circle with diameter BC intersect AC at D. Let the tangent to this circle at D intersect AB at E. What is the value of $\frac{AE}{BE}$?

3 Square $ABCD$ has side length 4. Points P and Q are located on sides BC and CD, respectively, such that $BP = DQ = 1$. Let AQ intersect DP at point X. Compute the area of triangle PQX.

4 Let $ABCD$ be a quadrilateral such that $AB = BC = 13$, $CD = DA = 15$ and $AC = 24$. Let the midpoint of AC be E. What is the area of the quadrilateral formed by connecting the incenters of ABE, BCE, CDE, and DAE?

5 Find the smallest possible number of edges in a convex polyhedron that has an odd number of edges in total has an even number of edges on each face.

6 Consider triangle ABC on the coordinate plane with $A = (2, 3)$ and $C = \left(\frac{96}{13}, \frac{207}{13}\right)$. Let B be the point with the smallest possible y-coordinate such that $AB = 13$ and $BC = 15$. Compute the coordinates of the incenter of triangle ABC.

7 Let ABC be an acute triangle with $BC = 4$ and $AC = 5$. Let D be the midpoint of BC, E be the foot of the altitude from B to AC, and F be the intersection of the angle bisector of $\angle BCA$ with segment AB. Given that AD, BE, and CF meet at a single point P, compute the area of triangle ABC. Express your answer as a common fraction in simplest radical form.

8 Consider an acute angled triangle $\triangle ABC$ with side lengths $7, 8$, and 9. Let H be the orthocenter of ABC. Let Γ_A, Γ_B, and Γ_C be the circumcircles of $\triangle BCH$, $\triangle CAH$, and $\triangle ABH$ respectively. Find the area of the region $\Gamma_A \cup \Gamma_B \cup \Gamma_C$ (the set of all points contained in at least one of Γ_A, Γ_B, and Γ_C).

9 Let ABC be a right triangle with hypotenuse AC. Let G be the centroid of this triangle and suppose that we have $AG^2 + BG^2 + CG^2 = 156$. Find AC^2.

10 Three circles with radii 23, 46, and 69 are tangent to each other as shown in the figure below (figure is not drawn to scale). Find the radius of the largest circle that can fit inside the shaded region.
`https://cdn.artofproblemsolving.com/attachments/6/d/158abc178e4ddd72541580958a4ee2348b202`
`png`

– Geometry Tiebreaker

1 Pentagon $ABCDE$ has $AB = BC = CD = DE$, $\angle ABC = \angle BCD = 108^o$, and $\angle CDE = 168^o$. Find the measure of angle $\angle BEA$ in degrees.

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2 On each edge of a regular tetrahedron, five points that separate the edge into six equal segments are marked. There are twenty planes that are parallel to a face of the tetrahedron and pass through exactly three of the marked points. When the tetrahedron is cut along each of these twenty planes, how many new tetrahedrons are produced?

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3 Three cities that are located on the vertices of an equilateral triangle with side length 100 units. A missile flies in a straight line in the same plane as the equilateral triangle formed by the three citiies. The radar from City A reported that the closest approach of the missile was 20 units. The radar from City B reported that the closest approach of the missile was 60 units. However, the radar for city C malfunctioned and did not report a distance. Find the minimum possible distance for the closest approach of the missile to city C.
